



UNIVERSIDADE FEDERAL DE UBERLÂNDIA

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**NONLINEAR BEHAVIOR INVESTIGATION OF TYPICAL
SECTION WITH STRUCTURAL NONLINEARITIES**

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ABSTRACT

In order to forecast LCO (Limit cycle oscillation) brought on by nonlinear features associated with the control surface degree of freedom, this work examines the nonlinear aeroelastic behavior of the control surface of a typical section. Since traditional techniques of aeroelastic stability solution, which rely on linear models, cannot foresee these effects, the aeroelastic stability investigation around the effects of structural nonlinearities has gained significant relevance in the field of aeroelasticity study. In this context, control surfaces with freeplay are widely used as a case study of the approaches provided in the literature around aeroelastic stability from the standpoint of representing real cases of unwanted nonlinear behavior that can lead the system to critical unstable conditions prior to unstable conditions predicted by linear models and solutions. For the purpose of predict and characterize LCO cases, this work uses frequency and time domain analyses, taking into account the simultaneous actions of dry friction, nonlinear damping, and freeplay on the system control surface. As a function of LCO characteristics obtained by the HM (Harmonic Balance method), the nonlinear functions are treated as linear equivalent properties for the frequency domain analysis. These properties are then associated with the linear aeroelastic model and traditional flutter solutions using the ELT (Equivalent Linearization Technique) process to develop the LCO map. The LCO map for the time domain analysis is derived from the use of the Fast Fourier Transform in time simulation samples involving a nonlinear aeroelastic model that is solved using a fourth-order Runge-Kutta solver in the Simulink software. The linear aeroelastic model and nonlinear aeroelastic model used for both processes are represented by state space equations with aerodynamic approximations done through the RFA (Rational Functions Approximation) method. In comparison to the dynamic system response observed in the time simulations, the results show that the RFA applied to frequency domain analysis produces an effective prediction of LCO characteristics in terms of frequency and amplitude. Additionally, depending on the imposed disturbance, the time domain results also demonstrate that the system can reach alternative LCO processes under the same conditions. Above all, the method worked satisfactorily to examine the LCO situations for an aeroelastic model.

Keywords: Nonlinear Aeroelasticity; Limit cycle oscillations; Nonlinear aeroelastic model.

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LIST OF ABBREVIATIONS AND ACRONYMS

AICs	Aerodynamic Influence Coefficients
DOF	Degree-of-freedom
ELT	Equivalent Linearization Technique
FFT	Fast Fourier Transform
HM	Harmonic Balance method
LCO	Limit cycle oscillation
RFA	Rational Functions Approximation

LIST OF SYMBOLS

α	Pitch displacement about the elastic center
β	Control surface (flap) displacement about the hinge line
δ	Freeplay width
θ	Angle of attack
ρ	Air density
ω	Oscillation frequency
$A_{(.)}$	Dynamic matrix
$A_{ae(.)}$	dynamic matrix in state space form
$B_{(.)}$	Viscous damping matrix
C_{nd}	Nonlinear Damping Coefficient
C	Friction Amplitude
F_{cn}	Cossine Function
F_{sn}	Sine Function
I	Moment of Inertia
$K_{(.)}$	Stiffness matrix
$M_{(.)}$	Inertia matrix
Q	Aerodynamic influence coefficients matrix
S	Static moment
T	Kinetic energy
U	LCO Amplitude
V_{∞}	Flow velocity
V	Potential energy
a	Distance from middle section to elastic center
b	Semi-chord
c	Distance from middle section to elastic center for β
f	Force and flutter(subscript)
g	Structural damping
h	Plunge Displacement
k	Reduced frequency
m	Total mass (per unit of span)
p	Eigenvalue
q_{∞}	Dynamic pressure
x_{DOF}	displacement vector
y_{DOF}	generic degree of freedom

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NOTATION

In this work, scalars are presented using normal weighted characters, e.g. $\alpha \in \mathbb{R}$, vectors are denoted with lower-case bold symbols, e.g. $\mathbf{x} \in \mathbb{R}^{N \times 1}$, and matrices are denoted using upper-case bold symbols, e.g. $\mathbf{K} \in \mathbb{R}^{N \times N}$.

1 INTRODUCTION

In a short amount of time, the aviation sector has advanced quickly in order to meet the demand for high-performance aircraft for both commercial and military transportation needs. Aeroelasticity is one of the most crucial aspects of aircraft design, regardless of the project that is being developed or modified.

The concept of aeroelasticity emerged in the aviation scene alongside the first motorized aircraft in the early 19th century, driven by the observation of adverse effects compromising aircraft performance in flight as well as the occurrence of unexpected catastrophic structural failures (GARRICK; REED III, 1981). From the earliest documented cases, aviation pioneers and their successors were dedicated not only to identifying the origins of these issues, but also to addressing them with solutions that ensured aircraft performance and safety.

The consolidation of airplane aeroelasticity came in between the 1950s and the 1970s with the development of high-speed aircraft (LIVNE, 2003). Several works proposed mathematical and experimental methods to model, understand, and predict static and dynamic aeroelasticity phenomena, transitioning from analytical methods for a few degrees of freedom to complete aircraft representation with computational methods (DUGUNDJI, 2003; ROSKAM, 2007). Among the instability events studied, there was a big concern to explore and characterize the flutter phenomenon, caused by an auto-excitation between two or more modes of vibration, that can cause catastrophic damage to the aircraft. To determine its characteristics in the frequency domain, the classical flutter solutions depend on linear mathematical models (WRIGHT; COOPER, 2008; HODGES; PIERCE, 2011; BISPLINGHOFF; ASHLEY; HALFMAN, 2013; CHUESHOV et al., 2015).

Despite the fact that linear methods are widely used and essential to understanding the dynamic aeroelastic aircraft behavior, these models do not consider nonlinearities. The presence of nonlinearities in structures, aerodynamics, and control systems can significantly impact aeroelastic behavior (DIMITRIADIS, 2017). In addition, nonlinear systems are the only ones where phenomena like LCO (Limit cycle oscillation) can occur. Particularly, the LCO is an oscillatory phenomenon that can lead to divergent dynamic instability in the system before the flutter predicted by linear solution methods (DIMITRIADIS, 2017).

Structural nonlinearities in control surfaces are a potential application field for studies about LCO phenomenon. There are several publications about this field, and the authors were concerned about developing models to include nonlinearities of systems in aeroelastic stability solutions, making possible the prediction of new critical points and the characterization of LCO (PRICE; ALIGHANBARI; LEE, 1995; CONNER et al., 1997; TANG; DOWELL, E. H.; VIRGIN, 1998; YANG; ZHAO, 1988; TANG; KHOLODAR, D.; DOWELL, E. H., 2000; ELLER, 2007; WAYHS-LOPES; DOWELL, E. H.; BUENO, 2023).

Among the methods used to include nonlinearities in aeroelastic models are numerical integration methods, which allow the observation of the system's response over time and, consequently, the behavior of the LCO phenomenon through the integration of the equations of motion with nonlinear expressions included. The advantage of numerical methods is the observation of the real oscillatory behavior caused by the structural nonlinearity of the system; however, they do not have any prior information about the stability of the system prior to the linear flutter speed.(PRICE; ALIGHANBARI; LEE, 1995; CONNER et al., 1997; TANG; DOWELL, E. H.; VIRGIN, 1998; YANG; ZHAO, 1988)

On the other hand, there are analytical methods, such as HM method and describing functions, that represent nonlinearities in aeroelastic models using equivalent linear functions. Such analytical methods do not consider the variation of initial conditions or disturbances, however, they are simple approximation methods that enable the use of traditional analytical solutions and the prior mapping of possible LCO points in terms of speed, amplitude, and frequency at speeds lower than those of linear flutter, which can be confirmed later by simulations over time (CONNER et al., 1997; LIU; DOWELL, E. H., 2005).

This work deals with a study of the case of a typical section with 3 DOF (Degree-of-freedom) with a trailing edge flap that includes freeplay, friction, and nonlinear damping acting simultaneously. The purpose is to present a process to obtain the LCO map due structural nonlinearities in the frequency domain, through a linear equivalent aeroelastic model and also in the time domain, through a nonlinear aeroelastic model.

1.1 DOCUMENT STRUCTURE

The document is structured as follows:

- Chapter 2: the literature review of linear and nonlinear aeroelasticity used as the theoretical and methodology bases of this study;
- Chapter 3: the aeroelastic modeling of the 3 DOF typical section. Also, the section will present the harmonic balance method applied for freeplay, friction, and nonlinear damping, modeling their integration with the aerolastic model. Finally, the time domain analysis process to verify the frequency domain LCO map;
- Chapter 4: the results of the frequency and time domain analysis considering all methodologies described in Chapter 3 applied;
- Chapter 5: the summary of the results and the discussion about the LCO characterization and the effects of nonlinearities in the aeroelastic system.

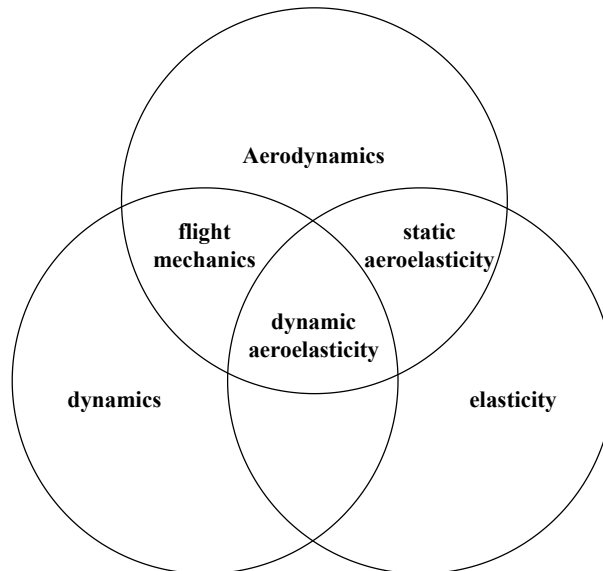
2 LITERATURE SURVEY AND THEORY

This chapter introduces the theory and literature research on the concepts used as a basis for developing the contribution of this application. The aeroelasticity concepts and the linear approaches used to tackle the dynamic instability problem (flutter) are reviewed in the first section. The second portion will provide a review of nonlinear aeroelasticity, going over the definition of LCO phenomenon. Finally, the harmonic balance method and its relationship to the linear model's eigenvalue solution will be covered in the third portion, as well as the process of time simulations with the nonlinear model.

2.1 LINEAR AEROELASTICITY

According to Wright and Cooper (2008) and Hodges and Pierce (2011), aeroelasticity can be defined as the study of the interaction among aerodynamic, inertial, and elastic forces within a flexible structure and the resultant phenomena. As illustrated in Figure 2.1, aeroelasticity has a multidisciplinary nature, and other major areas may arise from the interaction of two or all three of these forces. These fields were first described by Collar in the 1940s.

Figure 2.1 – Aeroelasticity diagram.



Source: Hodges and Pierce (2011).

In summary, aeroelasticity can be divided into two areas: static aeroelasticity and dynamic aeroelasticity. According to Wright and Cooper (2008), static aeroelasticity deals with non-oscillatory effects arising from the interaction between aerodynamic and elastic forces caused by aircraft flexibility. On the other hand, dynamic aeroelasticity focuses on the oscillatory effects resulting from these three forces.

In dynamic responses, movements involve significant accelerations and velocities, which introduce inertia components that interact with static and aerodynamic elements. Depending on its properties, an aircraft's dynamic response can be stable, neutral, or unstable.

The primary aeroelastic phenomenon associated with Dynamic Aeroelasticity and aeroelastic stability is flutter. In essence, flutter is a dynamic instability involving two or more flexible modes of aircraft vibration, i.e., it is a self-excitation phenomenon. This self-excitation is altered and intensified by extracting energy from the airstream. The amplitude of oscillation can grow exponentially, and this coupling system can become divergent, leading to catastrophic structural damage.

The flexible modes involved in the flutter problem could stem from different components of the aircraft or be related to the same component, such as wing bending/torsion, wing/engine, wing/control surface, or tail/control surface. The sources of excitation are diverse and can include the inherent dynamic behavior of the aircraft, influenced by dynamic pressure, mass distribution, and structural properties, as well as external factors like abrupt maneuvers or gust loads that excite the aircraft's elastic modes.

Linear aeroelastic systems are compatible with conventional flutter solutions. A typical way to think about a linear system is as a model made up of linear aerodynamic and structural models (DOWELL, E.; EDWARDS; STRGANAC, 2003).

2.2 NONLINEAR AEROELASTICITY

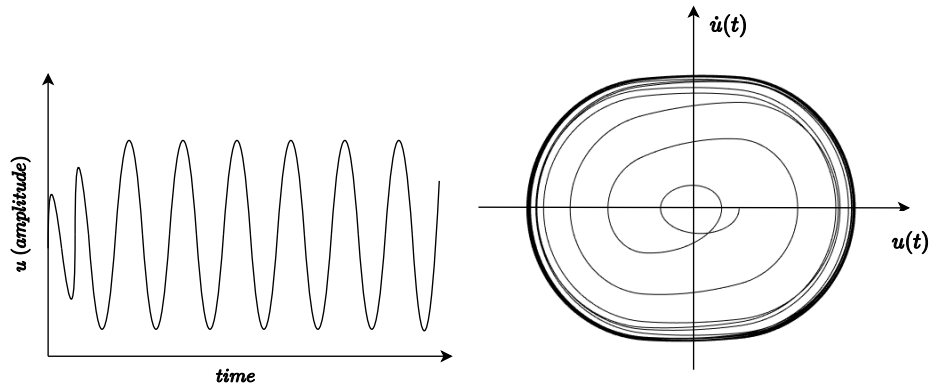
Aeroelastic systems present nonlinear characteristics, whose dynamic effects cannot be predicted using linear solutions and representations. In this sense, nonlinear aeroelasticity discusses theoretical and experimental approaches to including nonlinear functions in aeroelastic models and solutions (DOWELL, E.; EDWARDS; STRGANAC, 2003).

In addition, nonlinear aeroelasticity is concerned with predicting and characterizing the nonlinear dynamic response caused by these nonlinear characteristics. The LCO phenomenon sticks out among the nonlinear responses generated by the nonlinearities.

2.2.1 Limit-cycle oscillation (LCO)

The LCO (Limit cycle oscillation) is characterized by a steady-state oscillation with frequency and amplitude, as exemplified by Figure 2.2, characterized by the nonlinear system properties (HODGES; PIERCE, 2011). To capture this behavior, the aircraft must be treated as a nonlinear model, since LCO is a particular phenomenon of nonlinear systems (DOWELL, E.; EDWARDS; STRGANAC, 2003; DOWELL, E. H., 2021).

Figure 2.2 – LCO Representation.



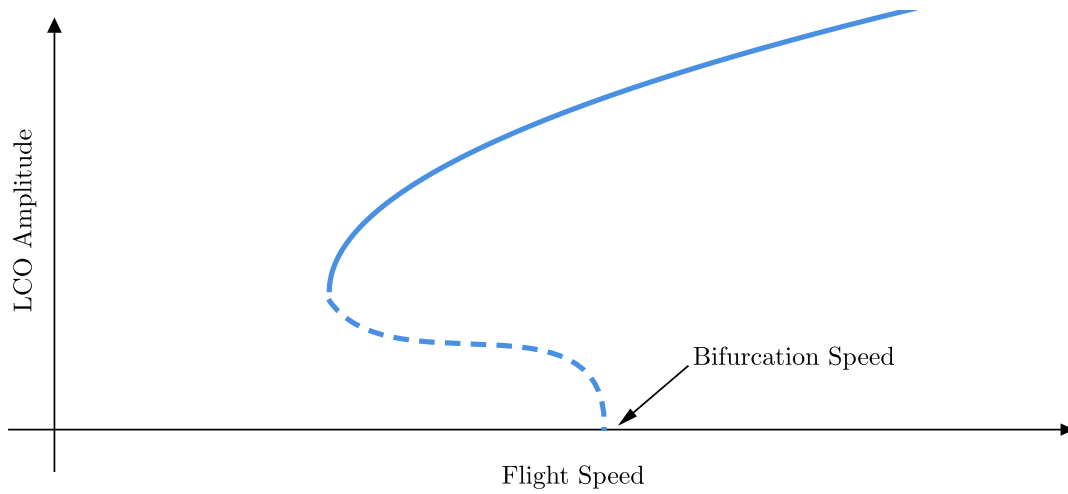
Source: Elaborated by the author.

LCO is not the only possible nonlinear aeroelastic behavior. However, through testing and studies, the general oscillatory behavior caused by nonlinearities can be characterized by LCO as a general description. The amplitude and frequency of LCO vary with flight parameters but are usually close to a frequency predicted by linear flutter analysis (DOWELL, E.; EDWARDS; STRGANAC, 2003).

Earl H Dowell (2021) explains that the LCO instability depends on its dynamic behavior due to different disturbances. Figure 2.3 depicts the amplitude of the LCO as a function of flight speed. The dotted line represents the unstable LCO while the solid line represents the stable LCO. Stable LCOs are those in which the motion returns to the same LCO mechanisms over time for any sufficiently small disturbance. Unstable LCOs are those in which the motion moves from its amplitude toward a stable, for any small disturbance. Both LCOs are dynamically conceivable if the system does not contain tiny perturbations (DOWELL, E. H.; TANG, 2002; DOWELL, E. H., 2021; KHOLODAR, D. B. et al., 2004).

The dotted and solid lines represented by Figure 2.3 are determined after a set of analyses. System response analysis or experimental data are required to define the LCO properties in terms of frequency and amplitude, to determine if these mechanisms are stable or not, and to know the sensibility of the system related to the magnitude of disturbances that move from a stable LCO mechanism to other stable LCO.

Figure 2.3 – Schematic LCO behavior.



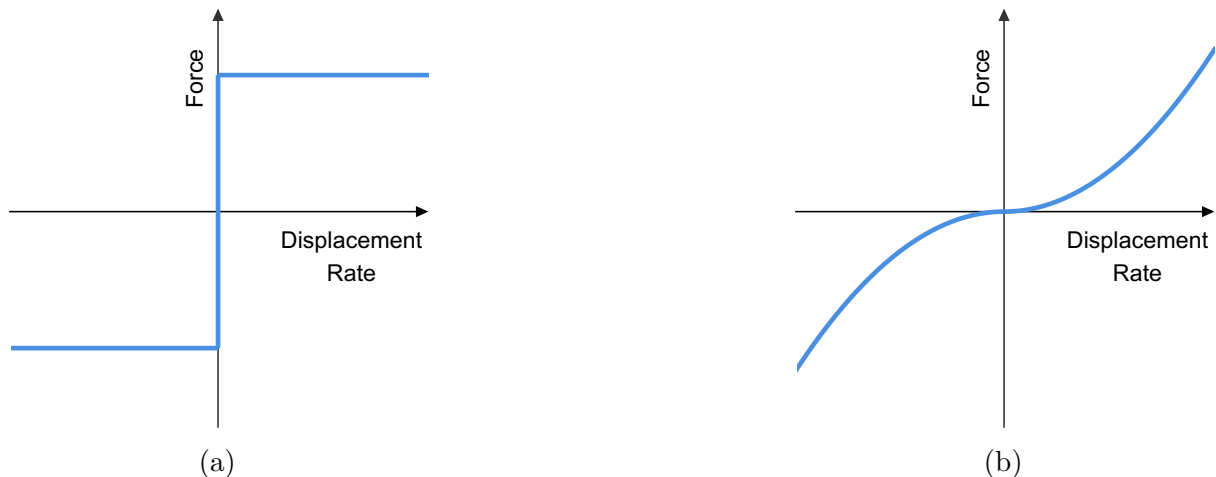
Source: Earl H Dowell and Tang (2002)

2.2.2 Nonlinearities Sources

Earl H Dowell (2021) and Dimitriadis (2017) bring together the three main sources of nonlinear functions that are frequently observed in aeroelastic systems: structural, aerodynamic, and control nonlinearities.

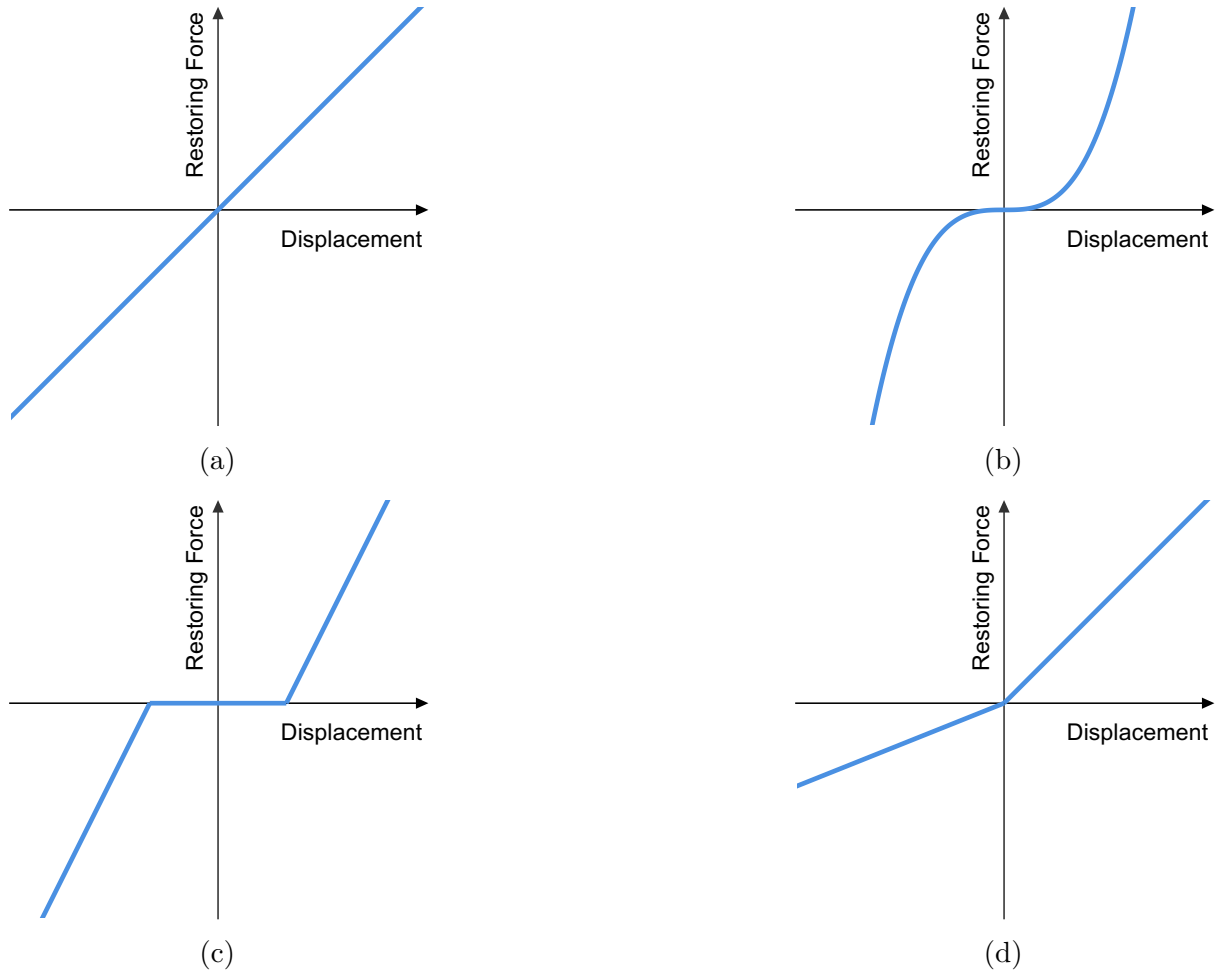
Structural nonlinearities can arise due to geometric reasons such as large deflections, nonuniform structural property effects, bilinearity, cubic stiffness, structural integrity partially lost as a freeplay, or due to dissipative nonlinearities like friction and other nonlinear damping forces. Figures 2.4 and 2.5 simplify structural nonlinearities.

Figure 2.4 – Structural nonlinearities in friction: (a) coulomb friction. (b) nonlinear friction.



Source: Elaborated by the author.

Figure 2.5 – Structural nonlinearities in stiffness: (a) linear stiffness. (b) cubic stiffness. (c) freeplay. (d) bilinear stiffness



Source: Elaborated by the author.

Aerodynamic nonlinearities can arise due to aerodynamic flows with transonic or unsteady regimes. Also can arise due to large and oscillating shocks or the combination with flow separation. Control nonlinearities can arise from multiple sources, such as actuator rate limits, deflection limits, and nonlinear control laws.

2.2.3 Harmonic Balance and the Equivalent Linearization Technique

Several numerical and analytical methods can be applied to include nonlinear functions in the aeroelastic models and solve their stability problem. The choice between these methods depends on the nonlinearity type, its nonlinear behavior, complexity, and computational time.

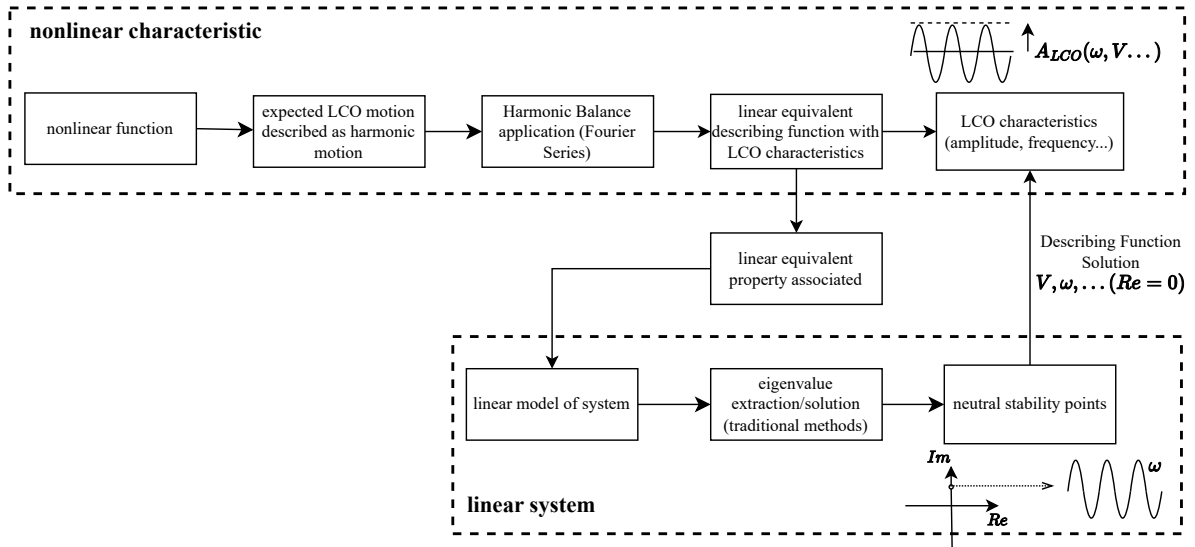
The HM (Harmonic Balance method) and the ELT (Equivalent Linearization Technique) have been used to predict limit cycle oscillations for aeroelastic systems, especially for structural nonlinearities, in the frequency domain due to the simplicity of the process compared with numerical methods and the possibility of using traditional methods of

aeroelastic stability solution (LIU; DOWELL, E. H., 2005; WAYHS-LOPES; DOWELL, E. H.; BUENO, 2023).

According to Figure 2.6, the nonlinear characteristic of a system, previously represented by a nonlinear function, passes by HM, which rewrites this expression as a linear equivalent describing function as dependent on LCO properties, amplitude, and frequency. This describing function can be represented by a linear equivalent parameter that will be connected to the linear system equations, forming a linear equivalent system.

From the solution of the stability problem of a linear equivalent system, using traditional methods, the neutral stability points, equivalent to oscillatory motions, will be used as the solution of the describing function. Finally, the properties of LCO will be determined.

Figure 2.6 – General application of HM (Harmonic Balance method) and the ELT (Equivalent Linearization Technique) to predict LCO in frequency domain



Source: Elaborated by the author.

The linear equivalent describing function $\hat{f}_{l_{eq}}$ is obtained by applying the Fourier series, according to Equations (2.1) and (2.2), with the nonlinear function f_{nl} described as a harmonic motion, single or multiple (DIMITRIADIS, 2017).

$$\hat{f}_{l_{eq}} = \frac{F_0}{2} + \sum F_{c_n} \cos(n\theta) + F_{s_n} \sin(n\theta) \quad (2.1)$$

$$\begin{aligned} F_{c_n} &= \frac{1}{\pi} \int_0^{2\pi} f_{nl} \cos(n\theta) d\theta \\ F_{s_n} &= \frac{1}{\pi} \int_0^{2\pi} f_{nl} \sin(n\theta) d\theta \end{aligned} \quad (2.2)$$

2.2.4 Previous Works - Structural Nonlinearities in control surfaces

Due to the fact that they are common characteristics of non-linear aeroelastic systems, structural nonlinearities in control surfaces are one of the most extensively studied problems in nonlinear aeroelasticity. Conner et al. (1997) was one of the pioneers in contributing to the investigation of LCO phenomenon, proposing nonlinear analyses in a 3DOF typical airfoil section with control surface freeplay. The authors determine the aeroelastic stability solution through a state space model that integrates the freeplay numerically, and the LCO characteristics are observed just in the time domain and in experimental cases.

Tang, Earl H Dowell, and Virgin (1998) proposed to analyze the LCO characteristics due to freeplay also in the frequency domain using a linearized described function using the harmonic balance method to assume the flap rotation as a single harmonic motion. Focusing on the dynamic response due to external forces, Tang, Denis Kholodar, and Earl H Dowell (2000) LCO study the airfoil section behavior due to freeplay excited by periodic gust loads in both frequency and time domains.

Together with Tang, Earl H Dowell, and Virgin (1998), there are many works that join the descriptive function from the harmonic balance method and the classic linear flutter solution (eigenvalue problem) to represent the nonlinear system and characterize the LCO phenomenon. Liu and Earl H Dowell (2005) expands the harmonic balance method with the adoption of more harmonics (high-order harmonic balance) to represent the nonlinear response.

The applications to predict LCO cases due to more than one nonlinearity also grow. Eller (2007) uses the harmonic balance with a single harmonic to model a control surface model with freeplay and Coulomb friction acting simultaneously in the system. Wayhs-Lopes, Earl H Dowell, and Bueno (2023) proposed a new method to analyze both nonlinearities, presenting an interactive approach in the frequency domain between describing functions and eigenvalue solutions.

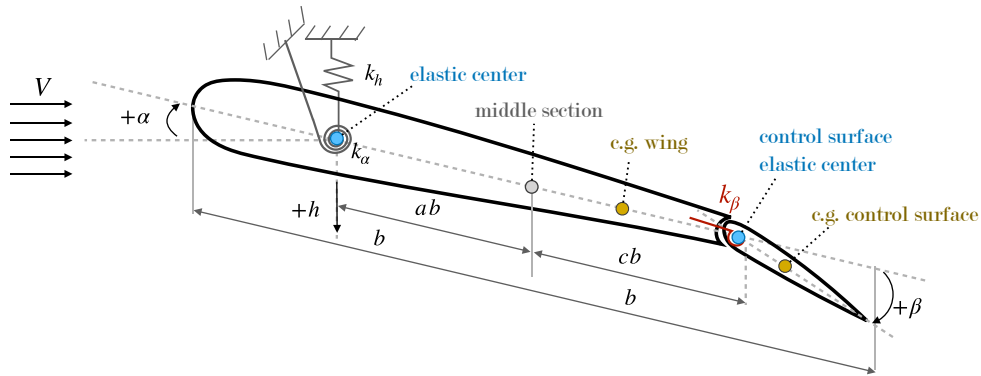
3 METHODOLOGY

This chapter covers the methodology applied. First, Section 3.1 addresses the equations of motion of the physical model, while Section 3.2 shows the aerodynamic forces depending on reduced frequency. Section 3.3 shows the traditional Flutter solution process based on the pk-method that will be used as the verification process of the state space aeroelastic model. Section 3.4 describes all steps of the development of the linear and nonlinear aeroelastic models through state space representation. Section 3.5 shows the linear equivalent describing functions for each nonlinear characteristic, obtained by HM (Harmonic Balance method). Section 3.6 contains the scheme of ELT (Equivalent Linearization Technique) interactive process that associates the linear equivalent describing functions to the linear model to generate the LCO map in frequency domain. Finally, Section 3.7 contains the scheme of numerical simulations with the nonlinear aeroelastic model to obtain the LCO from the dynamic response over time.

3.1 STRUCTURAL MODEL

The structural model is represented by a 3 DOF typical section with plunge (h), pitch (α) and control surface rotation (β), according to Figure 3.1, based on Wayhs-Lopes, Earl H Dowell, and Bueno (2023) and Liu and Earl H Dowell (2005). From a structural standpoint, the linear and torsional wing springs are located at the elastic center, where the wing bending and torsion forces are applied. The control surface spring is located at its hinge point, where the hinge moment is applied.

Figure 3.1 – 3DOF typical section physical model.



Source: Wayhs-Lopes, Earl H Dowell, and Bueno (2023).

The equations of motion are obtained through Lagrange's equation (HODGES; PIERCE, 2011),

$$\mathcal{L} = T - V \quad (3.1)$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right) - \frac{\partial \mathcal{L}}{\partial x_i} = 0 \quad (3.2)$$

where V denotes the potential energy, T denotes the kinetic energy, and x_i represents the generalized coordinates $x_i = [h(t), \alpha(t), \beta(t)]^T$. The kinetic and potential energy can be written as

$$T = \frac{1}{2}I_\alpha \dot{\alpha}^2 + \frac{1}{2}I_\beta \dot{\beta}^2 + \frac{1}{2}m\dot{h}^2 + (I_\beta + S_\beta b(c-a)) \dot{\alpha}\dot{\beta} + S_\alpha \dot{h}\dot{\alpha} + S_\beta \dot{h}\dot{\beta} \quad (3.3)$$

$$V = \frac{1}{2}K_h h^2 + \frac{1}{2}K_\alpha \alpha^2 + \frac{1}{2}K_\beta \beta^2 \quad (3.4)$$

where the time dependence notation (t) is omitted for better readability. Now, by replacing Equations (3.3) and (3.4) into (3.2) the differential equations of motion are obtained

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{h}} \right) - \frac{\partial \mathcal{L}}{\partial h} = (m\ddot{h} + S_\alpha \ddot{\alpha} + S_\beta \ddot{\beta} + K_h h) \quad (3.5)$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \right) - \frac{\partial \mathcal{L}}{\partial \alpha} = I_\alpha \ddot{\alpha} + (I_\beta + S_\beta b(c-a)) \ddot{\beta} + S_\alpha \ddot{h} + K_\alpha \alpha \quad (3.6)$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\beta}} \right) - \frac{\partial \mathcal{L}}{\partial \beta} = I_\beta \ddot{\beta} + (I_\beta + S_\beta b(c-a)) \ddot{\alpha} + S_\beta \ddot{h} + K_\beta \beta \quad (3.7)$$

Finally, by rearranging the Lagrange expressions and writing them in matrix form, the equation of motion is

$$\mathbf{M} \ddot{\mathbf{x}}_{DOF}(t) + \mathbf{K} \mathbf{x}_{DOF}(t) = \mathbf{0} \quad (3.8)$$

where,

$$\mathbf{x}_{DOF}(t) = \begin{Bmatrix} h(t) \\ \alpha(t) \\ \beta(t) \end{Bmatrix} \quad (3.9)$$

$$\mathbf{M} = \begin{bmatrix} m & S_\alpha & S_\beta \\ S_\alpha & I_\alpha & I_\beta + b(c-a)S_\beta \\ S_\beta & I_\beta + b(c-a)S_\beta & I_\beta \end{bmatrix} \quad (3.10)$$

$$\mathbf{K} = \begin{bmatrix} K_h & 0 & 0 \\ 0 & K_\alpha & 0 \\ 0 & 0 & K_\beta \end{bmatrix} \quad (3.11)$$

3.1.1 Viscous Damping

To develop the frequency domain LCO map, that will be detailed in Section 3.5, the friction and nonlinear damping effects are considered in the linear model as a linear equivalent viscous damping $\hat{d}_\beta$ proportional to displacement rate related to control surface DOF. The viscous damping is included as follows:

$$\mathbf{B} = \begin{bmatrix} 0 & & \\ & 0 & \\ & & \hat{d}_\beta \end{bmatrix} \quad (3.12)$$

Therefore, the equations of motion for the structural model including the linear equivalent viscous damping is

$$\mathbf{M}\ddot{\mathbf{x}}_{DOF}(t) + \mathbf{B}\dot{\mathbf{x}}_{DOF}(t) + \mathbf{K}\mathbf{x}_{DOF}(t) = \mathbf{0} \quad (3.13)$$

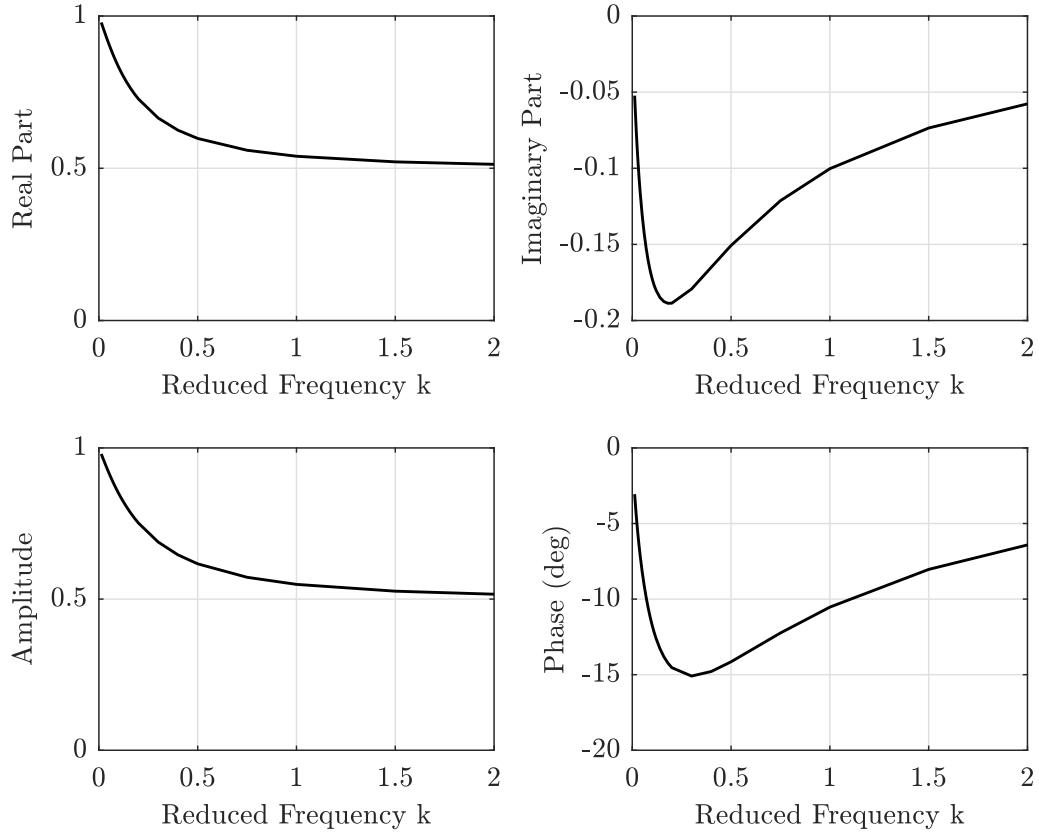
3.2 AERODYNAMIC MODEL

To compose the aeroelastic model, the aerodynamic loads must be written as a function of the structural model's generalized coordinates. Therefore, the aerodynamic force vector acting on the typical section is given by

$$\mathbf{f}_a(t) = \begin{Bmatrix} L(t) \\ M_\alpha(t) \\ M_\beta(t) \end{Bmatrix} \quad (3.14)$$

where $L(t)$ represents the lift force, $M_\alpha(t)$ the pitching moment, and $M_\beta(t)$ the control surface hinge moment. Henceforth, the time dependence notation will be omitted in all the following sections.

The solution of the flow around the airfoil is obtained through the unsteady aerodynamic theory and model proposed by Theodorsen (1949). This aerodynamic model is based on the assumption of thin airfoil-aileron undergoing small, simple harmonic oscillations. The aerodynamic loads present non-circulatory terms, dependent on the airfoil's states (position, velocity and acceleration), and circulatory terms dependent on the Theodorsen functions, $C(k)$, comprised by Figure 3.2.

Figure 3.2 – Theodorsen's Function $C(k)$.

Source: Elaborated by the author.

The variable k corresponds to the reduced frequency given by

$$k = \frac{\omega b}{V_\infty} \quad (3.15)$$

where b is the characteristic length of the model (semi-chord), V_∞ is the flow speed and ω is the airfoil frequency oscillation.

The aerodynamic force according to the Theodorsen (1949) is given by:

$$\mathbf{f}_a = \frac{\rho V_\infty^2}{2} \left(\frac{2b^2}{V_\infty^2} \mathbf{M}_{nc} \ddot{\mathbf{x}}_{DOF} + \frac{2b}{V_\infty} \mathbf{B}_{nc} \dot{\mathbf{x}}_{DOF} + 2\mathbf{K}_{nc} \mathbf{x}_{DOF} + 2C(k) \mathbf{K}_c \mathbf{x}_{DOF} + \frac{2V}{b} C(k) \mathbf{B}_c \dot{\mathbf{x}}_{DOF} \right) \quad (3.16)$$

where $\mathbf{M}_{nc}$, $\mathbf{K}_{nc}$ and $\mathbf{B}_{nc}$ are the aerodynamic noncirculatory terms and $\mathbf{K}_c$ and $\mathbf{B}_c$ the aerodynamic circulatory terms of lift, pitch moment and control surface moments relates to $\mathbf{x}_{DOF} = [h, \alpha, \beta]^T$.

Considering a simple harmonic motion imposed to $\mathbf{x}_{DOF} = \mathbf{x}_{DOF} \cdot e^{pt}$, where $p = i\omega$ and consequently $\omega i = \frac{Vik}{b}$, the aeroelastic model is represented by

$$(\mathbf{M}(p)^2 + \mathbf{B}(p) + \mathbf{K}) \cdot \mathbf{x}_{DOF} = q_\infty \mathbf{Q}(ik) \mathbf{x}_{DOF} \quad (3.17)$$

with the right hand-side representing the aerodynamic forces. The $\mathbf{Q}(ik)$ represents the AICs (Aerodynamic Influence Coefficients) matrix, with the expanded form is represented by

$$\mathbf{Q}(ik) = 2 \left(\mathbf{M}_{nc}(ik)^2 + (\mathbf{B}_{nc} + C(k)\mathbf{B}_c)(ik) + (\mathbf{K}_{nc} + C(k)\mathbf{K}_c) \right) \quad (3.18)$$

3.3 PK-METHOD

The p-k method is used to obtain the linear model aeroelastic stability solution in terms of velocity, frequency, and damping, where damping is considered a physical value.

Considering the problem,

$$\left(p^2 \mathbf{M} + p \mathbf{B} + \mathbf{K} - \frac{\rho V_\infty^2}{2} \mathbf{Q}(ik) \right) \mathbf{x}_{DOF} = \mathbf{0} \quad (3.19)$$

where $p = i\omega$ is assumed just for the aerodynamic portion, given by

$$\mathbf{Q}(ik) = \mathbf{Q}^R(ik) + \frac{\mathbf{Q}^I(ik)}{\omega} \quad (3.20)$$

Equation (3.19) solution can be determined through the determinant

$$|\mathbf{A} - p\mathbf{I}| = 0$$

where $\mathbf{A}$ is the aeroelastic dynamic matrix, described by

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1} \left(\mathbf{K} - \bar{q} \mathbf{Q}^R(ik) \right) & -\mathbf{M}^{-1} \left(\mathbf{B} - \frac{\rho b V}{4} \frac{\mathbf{Q}^I(ik)}{k} \right) \end{bmatrix} \quad (3.21)$$

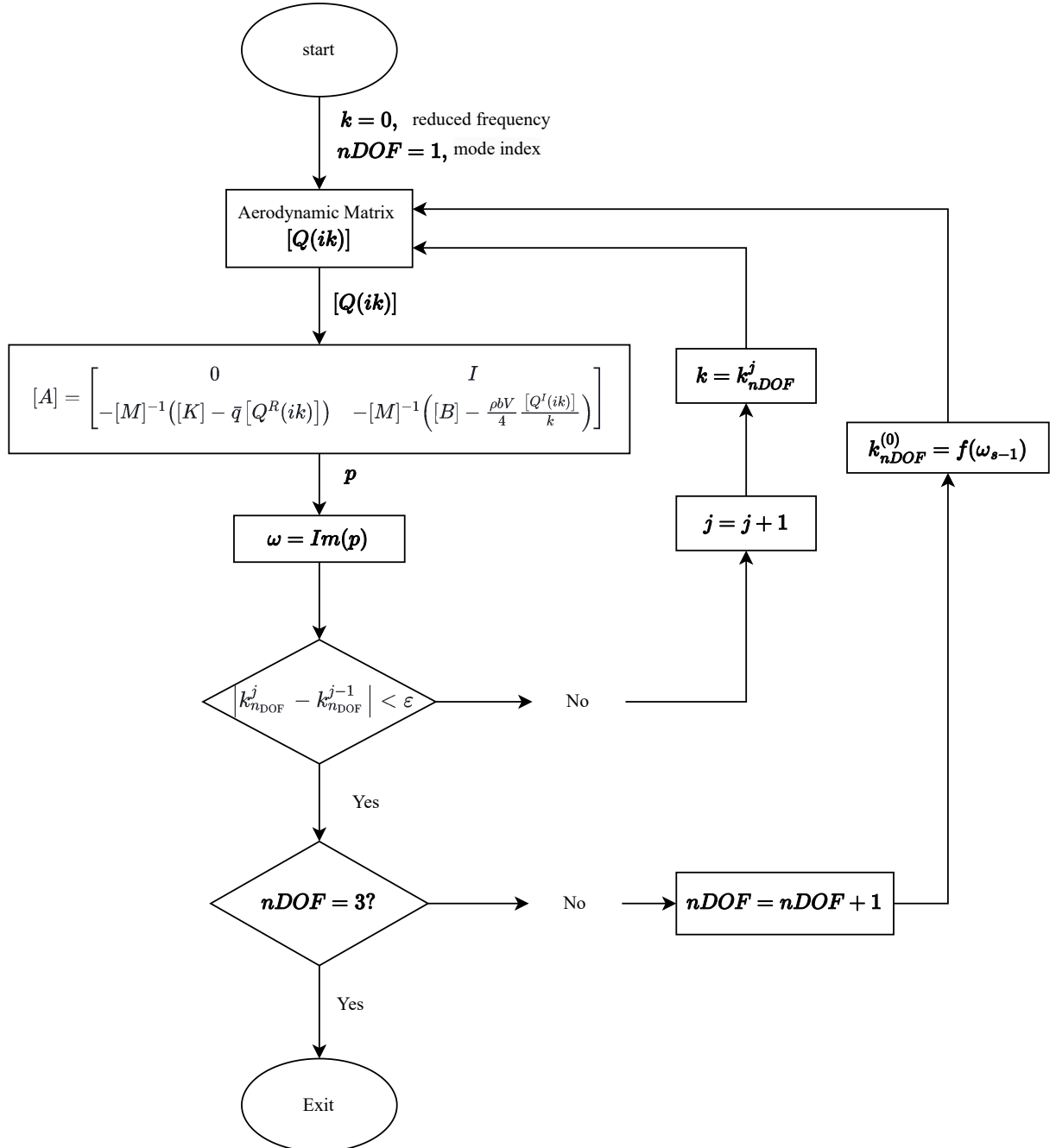
The determinant solution is a complex eigenvalue $p = \omega(\frac{g}{2} \pm i)$ that represents an oscillatory damped harmonic motion, where g is the modal damping in terms of structural damping.

As the aerodynamic model depends on a pure oscillatory motion $p = i\omega$ and the result of the eigenvalue problem is a complex eigenvalue, the p-k method consists of an interactive method, illustrated in Figure 3.3, based on works Silva et al. (2022), Hodges and Pierce (2011), Wright and Cooper (2008). Starting with an initial guess of reduced frequency k for calculating the AICs and subsequently solving the eigenvalue problem, the eigenvalue will only be considered satisfactory as a solution if the imaginary part of $p(\omega)$, considering a convergence criterion, corresponds to the value of k used in determining the aerodynamic loads, considering the correlation $\omega = kV/b$.

For a given velocity vector, the iterative process is applied to each degree of freedom of the aeroelastic model, storing the omega and damping values for each mode when the relationship between ω and k is met.

At the end of the velocity vector, the set of points is used to compose a v-g-f (velocity-frequency-damping) diagram of the model and determine the linear flutter velocity.

Figure 3.3 – p-k method application process



Source: Elaborated by the author.

3.4 STATE SPACE AEROELASTIC MODEL REPRESENTATION

The aeroelastic model is usually represented in the frequency domain due to the aerodynamic formulation (dependent on reduced frequency) and the methods used to solve the eigenvalue problem. However, it is not the only form of aeroelastic equation representation.

To analyze an LCO from a stability standpoint, time domain analysis is essential since the phenomenon depends on initial conditions or disturbances and the nonlinear characteristics must be included. For time domain aeroelastic analysis, concerned with predicting aircraft response behavior, a state space model is used to represent the nonlinear aeroelastic model.

3.4.1 Aerodynamic Approximation by Rational Functions

The main concern in developing the time domain aeroelastic system is to expand the AICs matrix $\mathbf{Q}(ik)$ from the frequency domain to the Laplace domain, making possible the integration of aerodynamic loads in the time domain. The methodologies and tools applied to it are based on Roger (1977), Wright and Cooper (2008), Technology (2017) and França (2022).

The aerodynamic AICs matrix in the Laplace domain can be written as a rational function given by

$$Q(s) = \mathbf{A}_0^{RFA} + \frac{b}{V_\infty} \mathbf{A}_1^{RFA} s + \frac{b^2}{V_\infty^2} \mathbf{A}_2^{RFA} s^2 + \sum_{n=1}^{N_L} \frac{s}{\left(s + \frac{V}{b} p_n\right)} \mathbf{A}_{n+2}^{RFA} \quad (3.22)$$

where the $\mathbf{A}_0^{RFA}$, $\mathbf{A}_1^{RFA}$ and $\mathbf{A}_2^{RFA}$ matrices represent the aerodynamic stiffness, aerodynamic damping, and aerodynamic apparent mass, respectively, while $\mathbf{A}_i^{RFA}$, $i = 3, 4, \dots, N_L, N_L + 1, N_L + 2$ represents the dynamic of lag due to the unsteady aerodynamic. N_L corresponds to the number of aerodynamic lag poles p_n .

The unknown $\mathbf{A}_i$, $i = 0, 1, 2, \dots, N_L, N_L + 1, N_L + 2$ matrices are obtained through RFA (Rational Functions Approximation) method application, proposed by Roger (1977), with the aim of determining the solution of $Q(ik)$ described as a sum of rational terms, as follows:

$$Q(ik) = A_0^{RFA} + A_1^{RFA} ik + A_2^{RFA} (ik)^2 + \sum_{n=1}^{N_L} \frac{ik}{(ik + p_n)} A_{n+2}^{RFA} \quad (3.23)$$

The solution is determined using the least square method given by

$$\begin{bmatrix} 1 & ik_1 & -k_1^2 & \frac{ik_1}{ik_1+p_1} & \cdots & \frac{ik_1}{ik_1+p_{N_L}} \\ 1 & ik_2 & -k_2^2 & \frac{ik_2}{ik_2+p_1} & \cdots & \frac{ik_2}{ik_2+p_{N_L}} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & ik_{N_k} & -k_{N_k}^2 & \frac{ik_{N_k}}{ik_{N_k}+p_1} & \cdots & \frac{ik_{N_k}}{ik_{N_k}+p_{N_L}} \end{bmatrix} \begin{bmatrix} A_O^{RFA} \\ A_1^{RFA} \\ A_2^{RFA} \\ \vdots \\ A_{N_L+2}^{RFA} \end{bmatrix}_{rs} = \begin{bmatrix} \mathbf{Q}(ik_1) \\ \mathbf{Q}(ik_2) \\ \vdots \\ \mathbf{Q}(ik_{N_k}) \end{bmatrix}_{as} \quad (3.24)$$

where $rs = (N_L + 3) \cdot N_{DOF} \times N_{DOF}$, $as = N_k \cdot N_{DOF} \times N_{DOF}$ and N_k is the number of reduced frequencies.

The objective of Equation (3.24) is to minimize the square error between the $\mathbf{Q}(ik)$ described in Section 3.2 by Equation (3.18) and the $\mathbf{Q}(ik)$ approximated by Rational Functions, given by

$$\varepsilon_{r_s} = \sum_{m=1}^{N_k} \left(\mathbf{Q}^{RFA}(ik_m) - \mathbf{Q}(ik_m) \right)^2, m = 1, 2, 3 \dots N_k \quad (3.25)$$

The lag poles values is given empirically by

$$p_n = -1.7k_{\max} \left(\frac{n}{N_L + 1} \right), n = 1, 2, 3 \dots N_L \quad (3.26)$$

where $k_{\max}$ is the highest reduced frequency adopted. The number of poles can be chosen arbitrarily without overtaking the number of reduced frequencies used in the approximation. A higher number of poles will reduce the least square error, at the cost of higher computational effort (TECHNOLOGY, 2017).

The present work considers seven aerodynamic poles/roots, defined by Equation (3.26). For state space representation, the aerodynamic lags augment the size of the aeroelastic problem so that

$$N = 2N_{DOF} + N_{DOF} N_L \quad (3.27)$$

where N_{DOF} is the number of structural DOF

3.4.2 Time Domain Aeroelastic Model Equations

The aeroelastic system with aerodynamics described in variable s can be represented as follows:

$$(\mathbf{M}s^2 + \mathbf{B}s + \mathbf{K})\mathbf{X}_{DOF}(s) = q_{\infty} \left(\mathbf{A}_0^{RFA} + \frac{b}{V} \mathbf{A}_1^{RFA} s + \frac{b^2}{V_{\infty}^2} \mathbf{A}_2^{RFA} s^2 + \sum_{n=1}^{N_L} \frac{s}{\left(s + \frac{V}{b} p_n\right)} \mathbf{A}_{n+1}^{RFA} \right) \mathbf{X}_{DOF}(s) \quad (3.28)$$

where $s = \frac{ikV}{b}$. For time domain aeroelastic model, Equation (3.28) can be rewritten so that

$$\bar{\mathbf{M}}\ddot{\mathbf{x}}_{DOF} + \bar{\mathbf{B}}\dot{\mathbf{x}}_{DOF} + \bar{\mathbf{K}}\mathbf{x}_{DOF} - q_\infty \sum_{n=1}^{N_L} \mathbf{x}_{a,n} = \mathbf{0} \quad (3.29)$$

where,

$$\bar{\mathbf{M}} = \mathbf{M} - q_\infty \frac{b^2}{V_\infty^2} \mathbf{A}_2^{RFA} \quad (3.30)$$

$$\bar{\mathbf{B}} = \mathbf{B} - q_\infty \frac{b}{V_\infty} \mathbf{A}_1^{RFA} \quad (3.31)$$

$$\bar{\mathbf{K}} = \mathbf{K} - q_\infty \mathbf{A}_0^{RFA} \quad (3.32)$$

The $\mathbf{x}_{a,n}$ are the so-called augmented states due to aerodynamic lags and are defined as

$$\mathbf{x}_{a,n} = \int_0^t \mathbf{A}_{n+2}^{RFA} e^{-\left(\frac{V_\infty}{b}\right) p_n(t-\tau)} \mathbf{x}_{DOF} d\tau, \quad n = 1, 2, 3..N_L \quad (3.33)$$

where $\dot{\mathbf{x}}_{a,n}$ is

$$\dot{\mathbf{x}}_{a,n} = \mathbf{A}_{n+2}^{RFA} \cdot \dot{\mathbf{x}}_{DOF} - \left(\frac{V_\infty}{b}\right) P_n \cdot \mathbf{x}_{a,n}(t) \quad (3.34)$$

3.4.3 Linear and Nonlinear State Space Aeroelastic Model

The formulation of the nonlinear aeroelastic state space model is modeled by the form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_{ae}\mathbf{x} + \mathbf{f}_{nl}(\mathbf{x}) \\ \tilde{\boldsymbol{\beta}} &= \mathbf{C}\mathbf{x} \end{aligned} \quad (3.35)$$

in which $\mathbf{x}$ is the $N \times 1$ state vector as follows

$$\mathbf{x} = \begin{Bmatrix} \mathbf{x}_{DOF} \\ \dot{\mathbf{x}}_{DOF} \\ \mathbf{x}_a \end{Bmatrix} \quad (3.36)$$

composed by the displacement and displacement rates of DOF system and the aerodynamic lag state. $\tilde{\boldsymbol{\beta}}$ are the vector with scalar values of angular displacement and displacement rate of control surface DOF, represented by

$$\tilde{\boldsymbol{\beta}} = \{\beta \quad \dot{\beta}\}^\top$$

The $\mathbf{A}_{ae}$ is the $N \times N$ aeroelastic dynamic matrix in state space, that represents the linearized system dynamics. The $\mathbf{A}_{ae}$ formulation is obtained with the combination between (3.29) and (3.34), generating $\mathbf{A}_{ae}$ according to Technology (2017):

$$\mathbf{A}_{ae} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ -\bar{\mathbf{M}}^{-1}\bar{\mathbf{K}} & -\bar{\mathbf{M}}^{-1}\bar{\mathbf{B}} & \bar{q}\bar{\mathbf{M}}^{-1}\mathbf{D} \\ \mathbf{0} & \mathbf{E} & \frac{V}{b}\mathbf{R} \end{bmatrix}_{N \times N} \quad (3.37)$$

where $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{R}$ are

$$\mathbf{D} = [\mathbf{I}, \mathbf{I}, \mathbf{I}, \dots, \mathbf{I}]_{(N_{DOF}) \times (N_{DOF} \cdot N_L)} \quad (3.38)$$

$$\mathbf{E} = \begin{bmatrix} \mathbf{A}_3^{RFA} \\ \mathbf{A}_4^{RFA} \\ \vdots \\ \mathbf{A}_{N_L}^{RFA} \end{bmatrix}_{(N_{DOF} \cdot N_L) \times N_{DOF}} \quad (3.39)$$

$$\mathbf{R} = \text{diag}(p_1, p_1, p_1, p_2, p_2, p_2, \dots, p_n, p_n, p_n)_{(N_{DOF} \cdot N_L) \times (N_{DOF} \cdot N_L)} \quad (3.40)$$

The function $\mathbf{f}_{nl}(\mathbf{x})$ represents the $N \times 1$ nonlinear functions of freeplay, friction, and nonlinear damping, $b^{\delta, C, C_{nd}}u$, where u is the scalar that represents $\beta(t) \pm \delta$ (freeplay), $\frac{\dot{\beta}(t)}{|\dot{\beta}(t)|}$ (friction), or $\dot{\beta}(t)|\dot{\beta}(t)|$ (nonlinear damping), so that

$$\mathbf{b}^{\delta} = \begin{bmatrix} 0 \\ -\bar{\mathbf{M}}^{-1}f^{\delta} \\ 0 \end{bmatrix}_{N \times 1}, \quad \mathbf{f}^{\delta} = \begin{bmatrix} 0 \\ 0 \\ K_{\beta} \end{bmatrix}_{N_{DOF} \times 1} \quad (3.41)$$

$$\mathbf{b}^C = \begin{bmatrix} 0 \\ -\bar{\mathbf{M}}^{-1}f^C \\ 0 \end{bmatrix}_{N \times 1}, \quad \mathbf{f}^C = \begin{bmatrix} 0 \\ 0 \\ C \end{bmatrix}_{N_{DOF} \times 1} \quad (3.42)$$

$$\mathbf{b}^{C_{nd}} = \begin{bmatrix} 0 \\ -\bar{\mathbf{M}}^{-1}f^{C_{nd}} \\ 0 \end{bmatrix}_{N \times 1}, \quad \mathbf{f}^{C_{nd}} = \begin{bmatrix} 0 \\ 0 \\ C_{nd} \end{bmatrix}_{N_{DOF} \times 1} \quad (3.43)$$

Matrix $\mathbf{C}$ is the $2 \times N$ output matrix, as follows

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \end{bmatrix}_{2 \times N} \quad (3.44)$$

The state space representation will also be used for frequency domain analysis, using the linear portion $\mathbf{A}_{ae}$ eigenvalue solution, and time domain analysis, using the nonlinear complete system defined by (3.35).

3.5 HARMONIC BALANCE APPLICATION

To develop the LCO maps in the frequency domain using the linear aeroelastic model represented by $\mathbf{A}_{ae}$, the nonlinear functions are linearized using the HM (Harmonic Balance method), which approximates the LCO behavior by a simple harmonic motion and generates linear equivalent properties that can be replaced on the stiffness and damping matrices. The next section details the application of HM process to obtain the linear equivalent parameters for freeplay, friction, and nonlinear damping.

The subscripts (δ) , (C) , (C_{nd}) represent the freeplay, friction, and nonlinear damping cases, respectively. The subscript (nl) indicates the nonlinear functions, while subscript (l_{eq}) represents the linear equivalent functions. The superscript $(\cdot)$ indicates the linear equivalent variables, nonlinear and linear functions with harmonic motion and Fourier series applied.

3.5.1 Freeplay

For the displacement DOF subject to a freeplay as the nonlinear system characteristic, Figure 3.4(a) represents the restoring force applied with a freeplay length denoted by 2δ , given by

$$f_{nl}^{\delta} = \begin{cases} K_{yDOF}(y_{DOF} + \delta) & , \quad y_{DOF} < -\delta \\ 0 & , \quad |y_{DOF}| \leq \delta \\ K_{yDOF}(y_{DOF} - \delta) & , \quad y_{DOF} > \delta \end{cases} \quad (3.45)$$

where y_{DOF} represent the displacement y imposed to the DOF of interest.

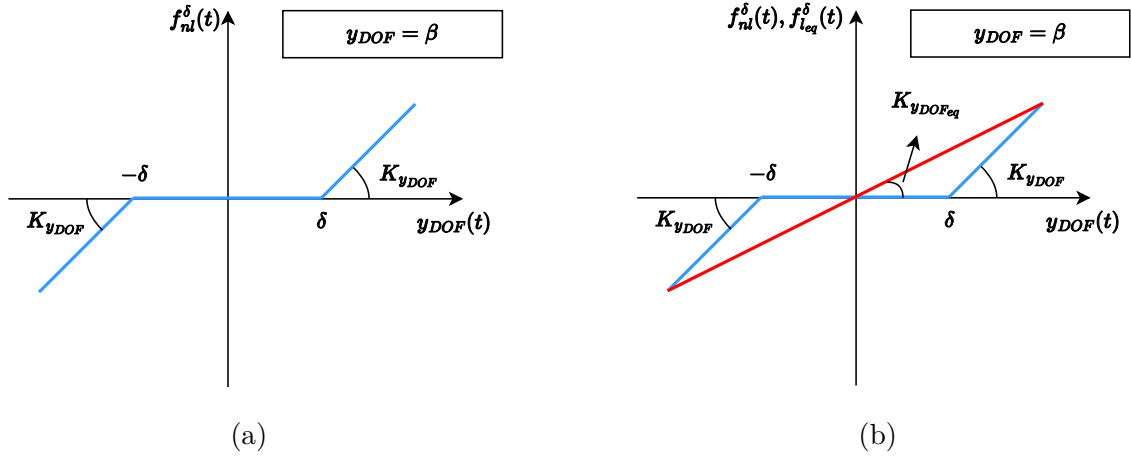
As illustrated by Figure 3.4(b), the HM aims to find a linear equivalent function that represents the effect of freeplay in the restoring force, given a linear equivalent stiffness less than K_{β} , that will be associated to the linear model, changing from

$$\mathbf{K} = \text{diag} (K_h, K_{\alpha}, K_{\beta})$$

to

$$\mathbf{K} = \text{diag} (K_h, K_{\alpha}, \hat{K}_{yDOF_{eq}})$$

Figure 3.4 – Representation of freeplay and linear equivalent functions: (a) freeplay (b) freeplay and linear equivalent curve.



Source: Elaborated by the author.

The application of HM depends on the type of harmonic motion expected by LCO behavior. From there, it is possible to stretch the nonlinear and linear equivalent forces caused by this periodic motion and define the functions and limits for Fourier series application.

3.5.1.1 Harmonic Motion

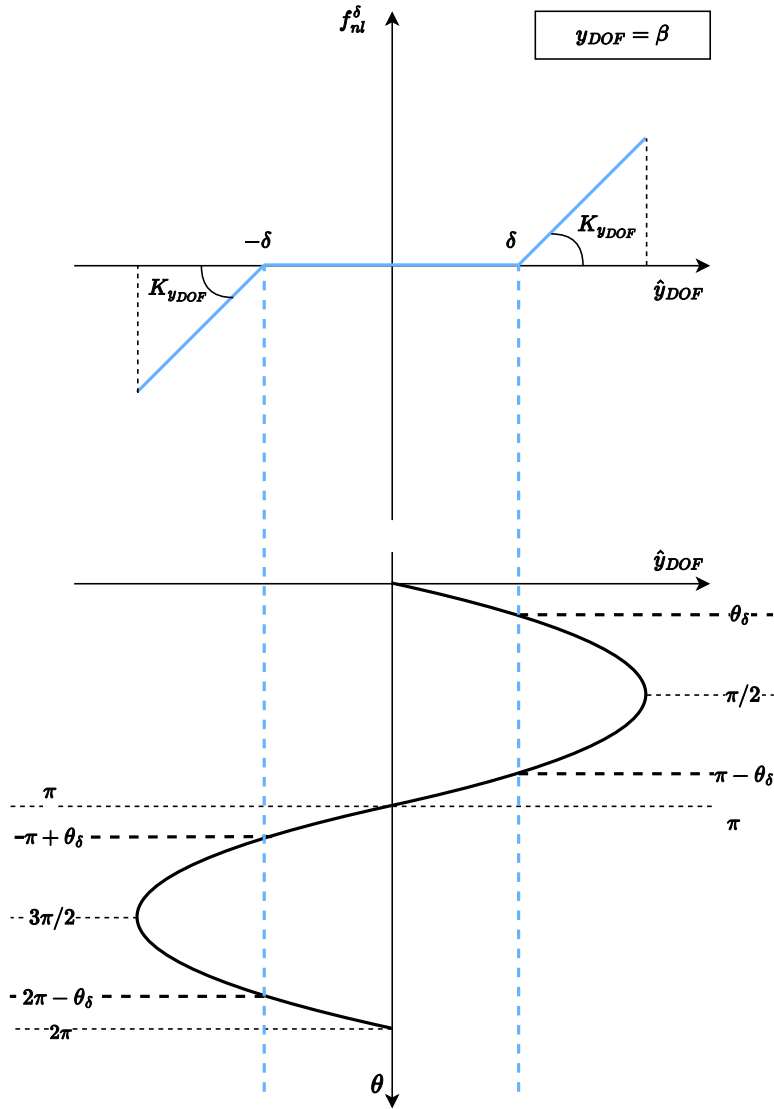
Considering a single harmonic motion $\hat{y}_{DOF} = U \sin \omega t$ in the control surface displacement to represent the LCO motion caused by freeplay, the linear equivalent force desired ($\hat{f}_{leq}$) is represented by

$$\hat{f}_{leq} = \hat{K}_{yDOF_{eq}} U \sin(\omega t) \quad (3.46)$$

where U represents the LCO amplitude, ω its frequency, and t is a time variable.

Considering $\omega t = \theta$, the nonlinear force depends on new limits. To understand, the LCO movement for one cycle is illustrated in Figure Figure 3.5 and compared with nonlinear freeplay. Considering the portions of motion where the restoring force is zero and replacing the new limits on (3.45), the nonlinear force with harmonic balance motion applied is characterized by Equation (3.47).

Figure 3.5 – Comparison between freeplay nonlinear function with one cycle of LCO motion assuming $\hat{y}_{DOF} = U \sin \theta$.



Source: Elaborated by the author.

$$f_{nl}^{\delta}(t) = \begin{cases} 0 & , & 0 < \theta \leq \theta_{\delta} \\ K_{yDOF}(U \cdot \sin(\theta) + \delta) & , & \theta_{\delta} < \theta \leq \pi - \theta_{\delta} \\ 0 & , & \pi - \theta_{\delta} < \theta \leq \pi + \theta_{\delta} \\ K_{yDOF}(U \cdot \sin(\theta) - \delta) & , & \pi + \theta_{\delta} < \theta \leq 2\pi - \theta_{\delta} \\ 0 & , & 2\pi - \theta_{\delta} < \theta \leq 2\pi \end{cases} \quad (3.47)$$

3.5.1.2 Fourier Series

To find the linear equivalent stiffness associated with LCO properties, the linear equivalent describing function is represented by the Fourier series, according to Equation (2.1) and Equation (2.2). Considering just the first harmonic ($n = 1$) to represent the LCO motion in (2.1) and replacing the terms of (3.47) in the nonlinear function (f_{nl}) in Equation (2.2), the linear equivalent describing function for freeplay is described by

$$\hat{f}_{l_{eq}}^\delta = F_{s_1} \sin(\theta) \quad (3.48)$$

where,

$$F_{s_1} = \frac{1}{\pi} \left(\int_{\theta_\delta}^{\pi - \theta_\delta} K_{yDOF} (U \cdot \sin(\theta) + \delta) d\theta + \int_{\pi + \theta_\delta}^{2\pi - \theta_\delta} K_{yDOF} (U \cdot \sin(\theta) - \delta) d\theta \right) \quad (3.49)$$

The solution of (3.48) and (3.49) yields

$$\hat{f}_{l_{eq}}^\delta = \frac{U K_{yDOF}}{\pi} \left[\frac{\pi}{2} - \theta_\delta - \sin \frac{(2\theta_\delta)}{2} \right] \sin(\theta) \quad (3.50)$$

where $\theta_\delta = \sin^{-1} \frac{\delta}{U}$. Now, by comparing (3.46) and (3.50), the equivalent stiffness is given by

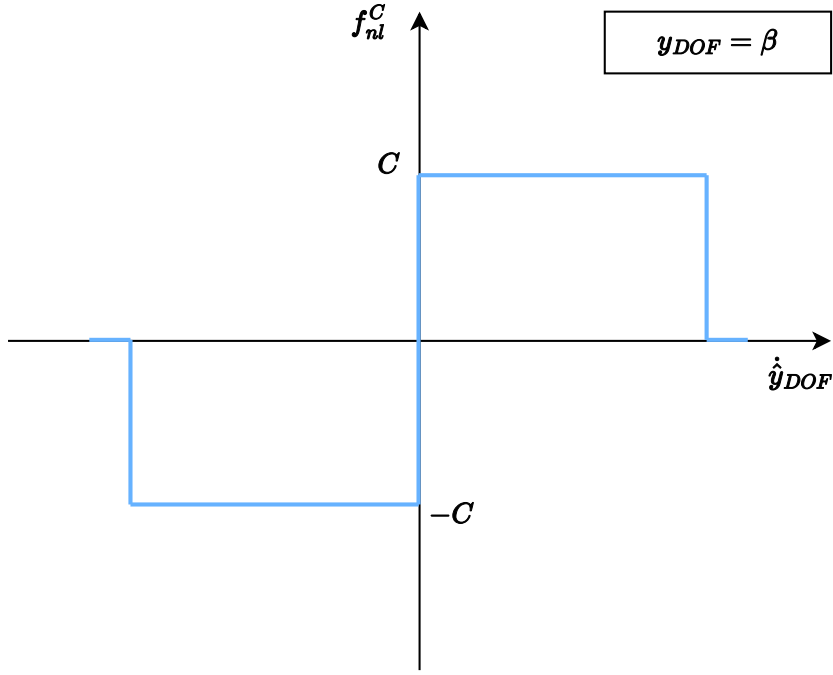
$$\hat{K}_{yDOF_{eq}} = \frac{K_{yDOF}}{\pi} \left[\frac{\pi}{2} - \theta_\delta - \sin \frac{(2\theta_\delta)}{2} \right] \quad (3.51)$$

(Intentionally left in blank)

3.5.2 Friction

The friction corresponds to a dissipative caused the displacement rate DOF as illustrated by Figure 3.6.

Figure 3.6 – Friction representation.



Source: Elaborated by the author.

The nonlinear function is represented by

$$f_{nl}^C(t) = \begin{cases} -C & , \dot{y}_{DOF} < 0 \text{ and } |y_{DOF}| > \delta \\ 0 & , |y_{DOF}| < \delta \\ C & , \dot{y}_{DOF} > 0 \text{ and } |y_{DOF}| > \delta \end{cases} \quad (3.52)$$

where C correspond to the friction amplitude.

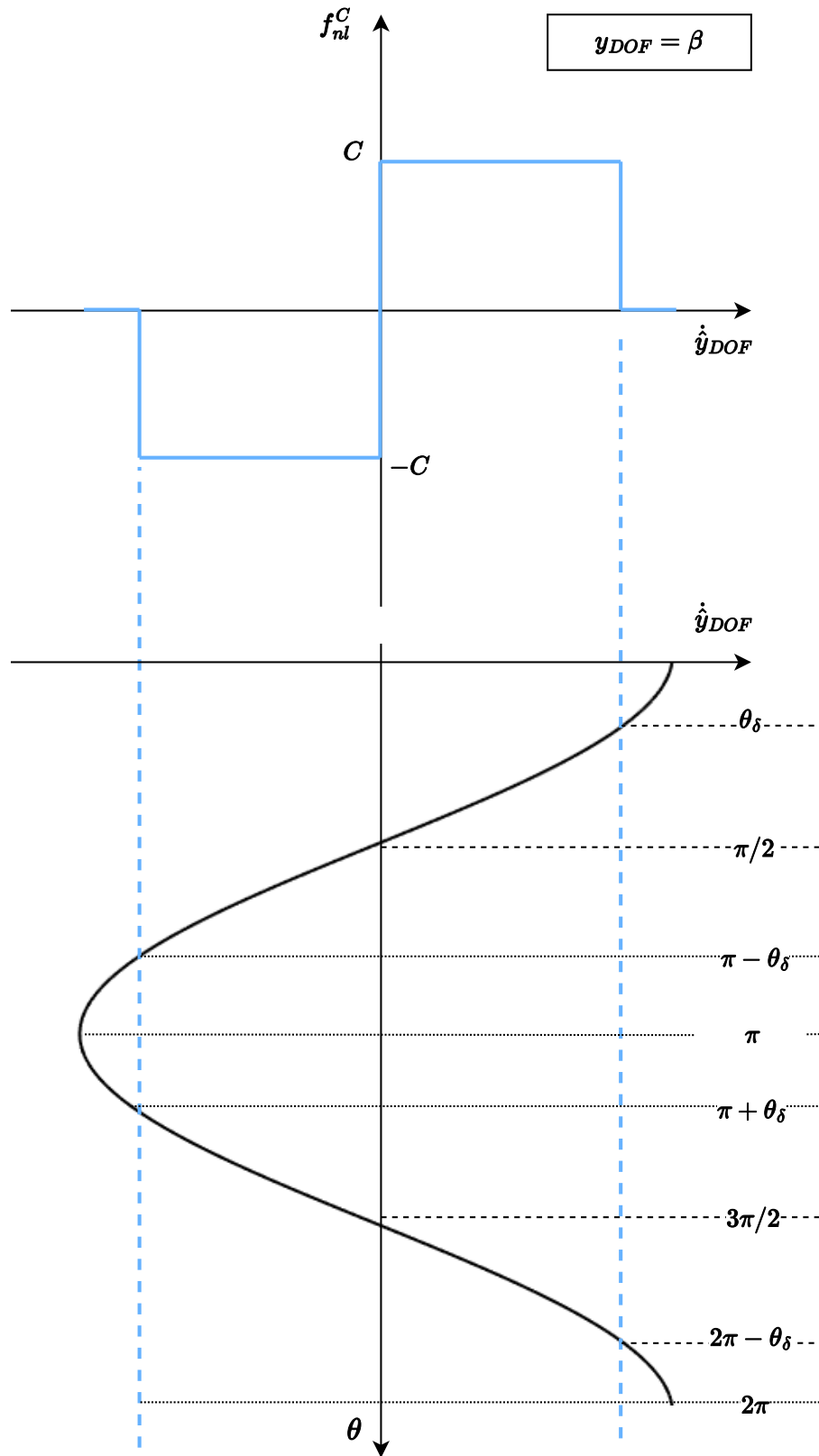
3.5.2.1 Harmonic Motion

Similarly to freeplay, the contribution in LCO due friction is modeled as a single harmonic motion $\hat{y}_{DOF}(t) = U \sin \omega t$, where $\hat{y}_{DOF} = U \omega \cos(\omega t)$. That way, the linear equivalent function desired ($\hat{f}_{leq}^C$) is represented by

$$\hat{f}_{leq}^C = \hat{d}_{y_{DOF_{eq}}} U \omega \cos(\omega t) \quad (3.53)$$

The comparison between LCO movement assuming $\dot{\hat{y}}_{DOF}(t) = U \omega \cos \theta$ for one cycle and compered with nonlinear function is illustrated in Figure Figure 3.7.

Figure 3.7 – Comparison between friction nonlinear function with one cycle of LCO motion assuming $\dot{y}_{DOF} = U\omega \cos \theta$.



Source: Elaborated by the author.

Thus, considering the new limits, the nonlinear function of friction yields

$$\hat{f}_{nl}^C(t) = \begin{cases} 0 & , & 0 < \theta \leq \theta_\delta \\ C & , & \theta_\delta < \theta \leq \frac{\pi}{2} \\ -C & , & \frac{\pi}{2} < \theta \leq \pi - \theta_\delta \\ 0 & , & \pi - \theta_\delta < \theta \leq \pi + \theta_\delta \\ -C & , & \pi + \theta_\delta < \theta \leq \frac{3\pi}{2} \\ C & , & \frac{3\pi}{2} < \theta \leq 2\pi - \theta_\delta \end{cases} \quad (3.54)$$

3.5.2.2 Fourier Series

Also considering just the first harmonic ($n = 1$) to represent the LCO motion in (2.1) and replacing the terms of (3.54) in the nonlinear function f_{nl} in Equation (2.2), the linear equivalent describing function for friction is described by

$$\hat{f}_{l_{eq}}^C = F_{c1} \cos(\theta) \quad (3.55)$$

where,

$$F_{c,1} = \frac{1}{\pi} \left(\int_{\theta_\delta}^{\pi/2} C \cos \theta d\theta + \int_{\pi/2}^{\pi-\theta_\delta} -C \cos \theta d\theta + \int_{\pi+\theta_\delta}^{3\pi/2} -C \cos \theta d\theta + \int_{3\pi/2}^{2\pi-\theta_\delta} C \cos \theta d\theta \right) \quad (3.56)$$

The solution of (3.55) and (3.56) yields

$$\hat{f}_{l_{eq}}^C = \frac{4C}{\pi} \left(1 - \frac{\delta}{U} \right) \cos \theta \quad (3.57)$$

Now, by comparing (3.53) and (3.57), the equivalent viscous damping due to freeplay is given by

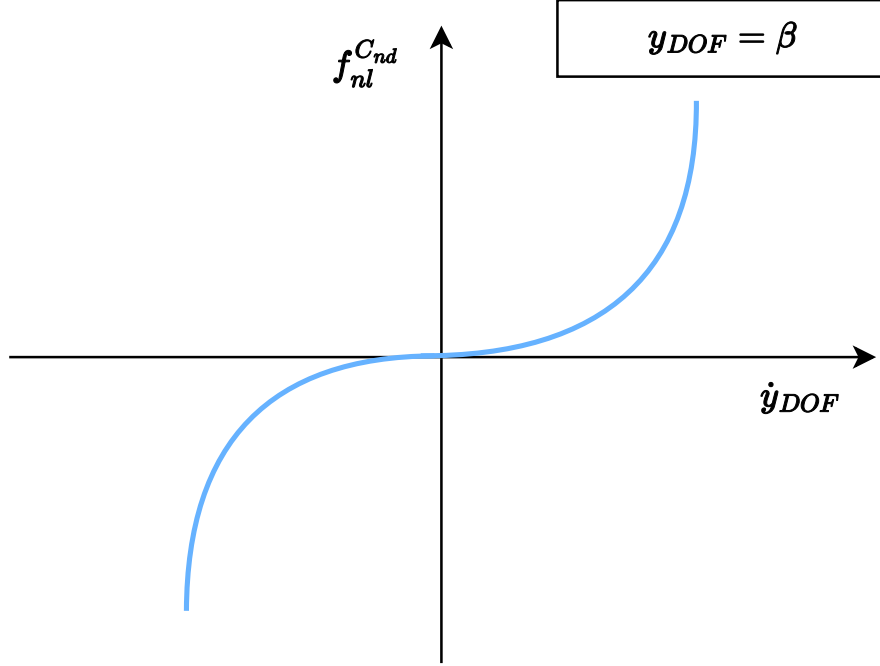
$$\hat{d}_{yDOFeq}^C = \frac{4C}{\omega U \pi} \left(1 - \frac{\delta}{U} \right) \quad (3.58)$$

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3.5.3 Nonlinear Damping

Lastly, nonlinear damping is considered. The nonlinear damping force is proportional to the square of the displacement rate, according to Figure 3.8.

Figure 3.8 – Damper nonlinear force representation.



Source: Elaborated by the Author.

The nonlinear function illustrated in Figure 3.8 is described as

$$f_{nl}^{C_{nd}} = C_{nd} |\dot{y}_{DOF}| \dot{y}_{DOF} \quad (3.59)$$

where C_{nd} represents the nonlinear damping coefficient.

Similarly, the damper nonlinear effect is approximated as a linear equivalent viscous damping associated with the LCO properties through the HM.

3.5.3.1 Harmonic Motion

Similarly to friction, the contribution in LCO due nonlinear damper is modeled as a single harmonic motion $\hat{y}_{DOF}(t) = U \cos(\omega t)$. That way, the linear equivalent function desired ($\hat{f}_{l_{eq}}^{C_{nd}}$) is represented by

$$f_{l_{Pq}}^{C_{nd}} = \hat{d}_{y_{DOF}}^{C_{nd}} \cdot U \omega \cos(\omega t) \quad (3.60)$$

Considering $\omega t = \theta$ and assuming the LCO movement by $\hat{y}_{DOF}(t) = U \cos \theta$ in (3.59) for one complete cycle, the nonlinear force assumes new limits, resulting in

$$\hat{f}_{nl}^{C_{nd}} = C_{nd} |U\omega \cos(\theta)| U\omega \cos(\theta), \quad 0 < \theta < 2\pi \quad (3.61)$$

3.5.3.2 Fourier Series

The linear equivalent viscous damping is also determined by a linear equivalent describing function, which is the Fourier series, where the linear equivalent describing function for friction is described by

$$\hat{f}_{l_{eq}}^{C_{nd}} = F_{c_1} \cos(\theta) \quad (3.62)$$

$$F_{c,1} = \frac{1}{\pi} \int_0^{2\pi} [C_{nd} |U\omega \cos(\theta)| U\omega \cos(\theta)] \cos \theta d\theta \quad (3.63)$$

The solution of (3.62) and (3.63) yields

$$\hat{f}_{l_{eq}}^{C_{nd}} = \frac{8C_{nd}U^2\omega^2 \cos \theta}{3\pi} \quad (3.64)$$

where the linear equivalent viscous damping is related to the nonlinear damper. The comparison between (3.60) and (3.64) is given by

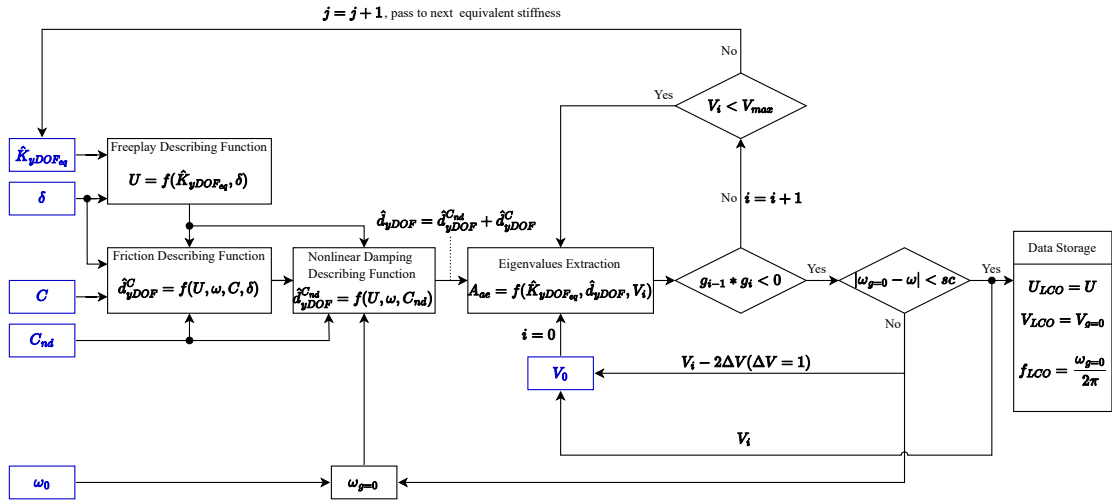
$$\hat{d}_{y_{DOFeq}}^{C_{nd}} = \frac{8C_{nd}U\omega}{3\pi} \quad (3.65)$$

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3.6 FREQUENCY DOMAIN ANALYSIS ROUTINE

The LCO map in the frequency domain is obtained through the interactive process proposed by Wayhs-Lopes, Earl H Dowell, and Bueno (2023) and illustrated by Figure 3.9. To obtain the LCO amplitude and frequency for each nonlinearity in the control surface, the process consists of matching the LCO properties of the three describing functions obtained in the previous section with the neutral stability points of the flutter solution, i.e., the characteristics of the aeroelastic system when the system crosses the imaginary axes.

Figure 3.9 – Interactive process to relate the describing functions with the ELT (Equivalent Linearization Technique) for all nonlinearities (freeplay, friction and nonlinear damping).



Source: Elaborated by the author.

The $\hat{K}_{yDOF_{eq}}$, δ , C , C_{nd} , ω_0 and V_0 values are the process inputs. Considering a vector of $\hat{K}_{yDOF_{eq}}$ values, with $0 < \hat{K}_{yDOF_{eq}} < K_\beta$, the process is done to each case of $\hat{K}_{yDOF_{eq}}$ defined within the range.

The process starts with freeplay. For the first $\hat{K}_{yDOF_{eq_j}}$ value, the LCO amplitude U is determined by the freeplay describing function, considering the freeplay width as 2δ . This amplitude is obtained and the freeplay width are replaced in friction and nonlinear damping describing functions considering ω_0 , determining the linear viscous damping. With the $\hat{K}_{yDOF_{eq_j}}$ and $\hat{d}_{yDOF_{eq}}$ defined, the ELT application starts.

The $\hat{K}_{yDOF_{eq_j}}$ and $\hat{d}_{yDOF_{eq}}$ values are replaced in the $\mathbf{A}_{ae}$ matrix (Equation (3.37)), considering the stiffness and damping matrices given by

$$\mathbf{K} = \text{diag} \left(K_h, K_\alpha, \hat{K}_{yDOF_{eq}} \right)$$

$$\mathbf{B} = \text{diag} \left(0, 0, \hat{d}_{yDOF_{eq}} \right)$$

Then, $\mathbf{A}_{ae}$ eigenvalues are extracted for each step velocity V_i , where $V_0 < V_i < V_{max}$, being V_0 and V_{max} as the start and maximum speed considering the flight envelope. To check if the system reached a point of neutral stability over the velocity range, the real part of each eigenvalue $p(V_i)$ is compared with the eigenvalue $p(V_i - 1)$. When $real(p(V_i))$ crosses the imaginary axes, i.e.,

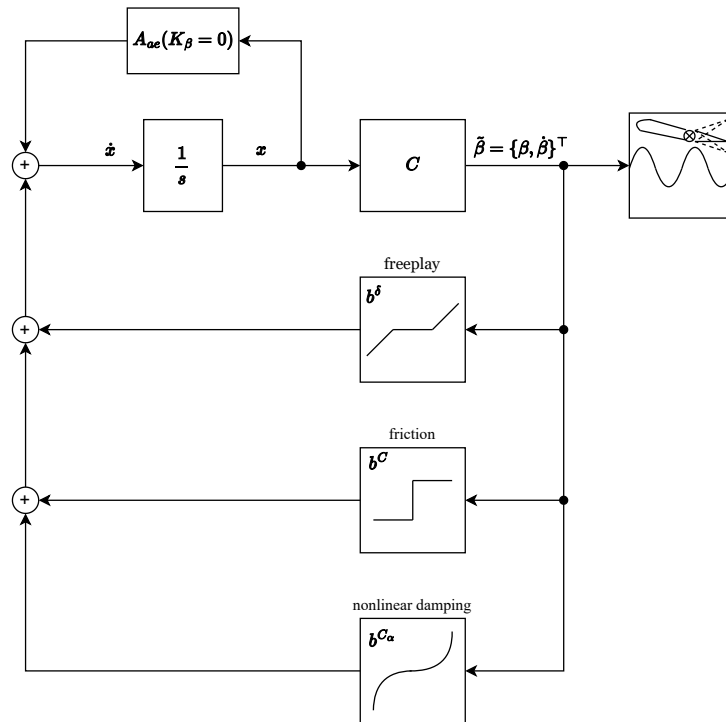
$$g_{i-1} \cdot g_i < 0$$

The point is considered a neutral stability point ($g = 0$). By a linear interpolation, the velocity and frequency $\omega_{g=0}$ related to the unstable point are calculated. If the difference between the $\omega_{g=0}$ and ω met the stop criteria, the point is considered a LCO case, and its properties are storage, continuing the process until $V_i = V_{max}$. If the stop criteria is not met, ω is set to $\omega = \omega_{g=0}$ and the starting velocity is set to $V_0 = V_i - 2\Delta V$ and the process is repeated. When the whole range of velocities is analyzed, the ELT finishes, and the whole process starts again for a new $\hat{K}_{yDOFeq_j}$.

3.7 TIME DOMAIN ANALYSIS ROUTINE

The LCO map in the time domain is obtained through time simulations illustrated by Figure 3.10. The nonlinear differential equations given by Equation (3.35) are solved numerically using Simulink, using a fourth-order Range-Kutta solver. To observe the LCO, a nonzero initial condition β_0 is considered.

Figure 3.10 – Representation of time simulation for nonlinear model.



Source: Elaborated by the author.

The linear stiffness and viscous damping properties of the control surface DOF in $\mathbf{A}_{ae}$ are not considered, and its nonlinear dynamics are included by the nonlinear functions $b^{\delta, C, C_{nd}} f$, acting simultaneously.

The time simulation is applied for the same range of velocities used by the frequency domain routine. From the control surface displacement signal in time, the LCO frequency and amplitude are calculated using the FFT (Fast Fourier Transform).

4 RESULTS

This chapter presents the results of the methodology developed in Chapter 3. Section 4.1 shows the results of the traditional method of flutter solution applied at the linear aeroelastic model, with the aim of knowing the linear flutter characteristics with and without K_β at the control surface DOF. Section 4.2 shows the results of aerodynamic approximation by RFA and the $\mathbf{A}_{ae}$ matrix verification through the comparison between its eigenvalue extraction and the pk-method solution. Section 4.3 shows the results of ELT interactive process using the describing function for each nonlinear characteristic, resulting in a LCO map in frequency domain dependent on freeplay width δ . Finally, section 4.4 shows the results given the time domain simulations using the nonlinear state space model and the comparison with the frequency domain map.

Table 1 shows the parameters of the aeroelastic system, as well as the values of the friction and non-linear damping constants used in the application case.

Table 1 – Physical and geometric parameters for the typical section

Variable	Symbol	Value	Unit
Span	s	0.50	m
Semi-chord	b	0.15	m
Total mass (per unit of span)	m	7.51	kg/m
Non-dimensional distance (w.r.t b) from middle section to elastic center (e.c)	a	-0.40	-
Non-dimensional distance (w.r.t b) from from middle section to control surface e.c.	c	0.60	-
Moment of Inertia (per unit of span) of wing-aileron around e.c	I_α	$4.77 \cdot 10^{-2}$	(kg · m ²)/m
Moment of Inertia (per unit of span) of control surface around e.c.	I_β	3.64×10^{-4}	(kg · m ²)/m
Static moment (per unit of span) of wing-control surface around e.c.	S_α	2.76×10^{-1}	(kg · m)/m
Static moment (per unit of span) of aileron around control surface e.c.	S_β	1.12×10^{-2}	(kg · m)/m
Stiffness in plunge h (per unit of span)	K_h	2669.12	(N/m)/m
Stiffness in pitch α (per unit of span)	K_α	188.47	(N/rad)/m
Stiffness in β (per unit of span)	K_β	2.82	(N/rad)/m
Friction Coefficient	C	1.25×10^{-3}	(N · m)/m
Nonlinear Damping Coefficient	C_{nd}	3.30×10^{-4}	(Nm · s ²)/(m · rad ²)

4.1 LINEAR FLUTTER SOLUTION

Freeplay is the main source of LCO since friction and nonlinear damping act as energy dissipatives in the system. The freeplay can be understood as the absence or decrease of equivalent stiffness in β . Therefore, it is interesting to analyze the effects of

variations in stiffness in β on the linear flutter solution since the results will be related to the development of the LCO map in the frequency domain.

Using the parameters of Table 1 on the linear aeroelastic model of Equation (3.17), Figure 4.1 shows the flutter solution obtained by pk-method process, treating the first case $\mathbf{K}$ as

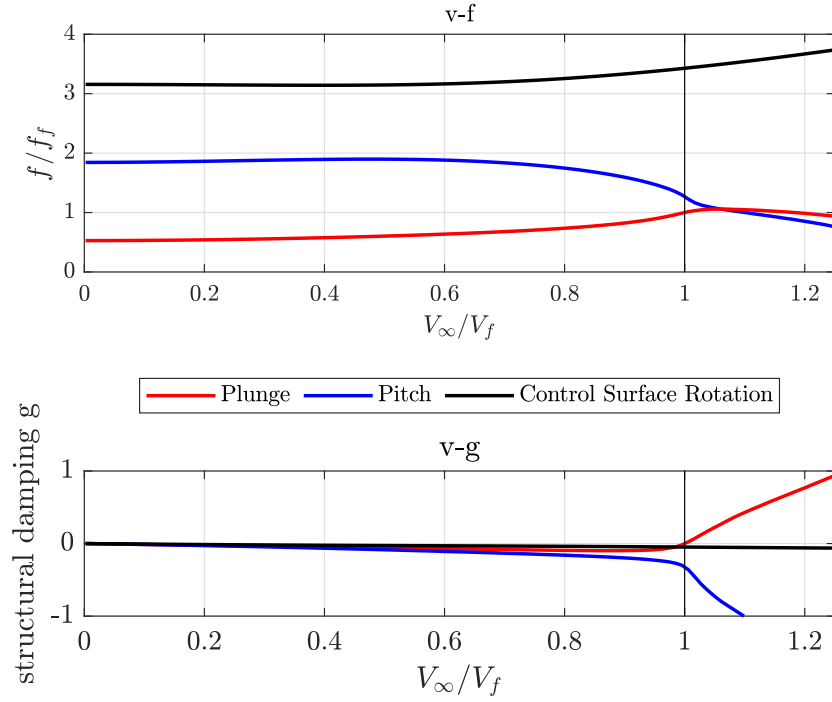
$$\mathbf{K} = \text{diag} \left(K_h, K_\alpha, K_\beta \right)$$

and the second case as

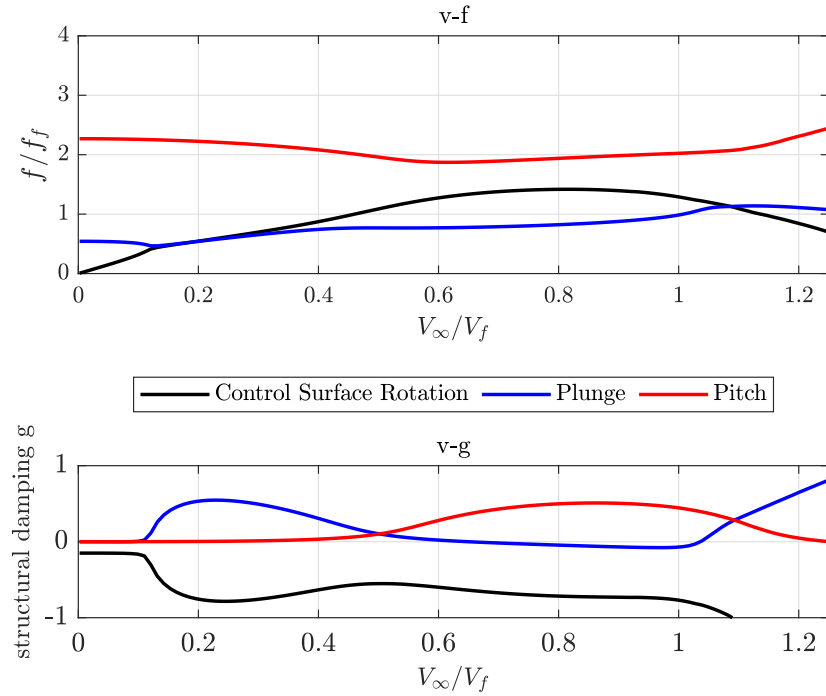
$$\mathbf{K} = \text{diag} \left(K_h, K_\alpha, 0 \cdot K_\beta \right)$$

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Figure 4.1 – vgf graph of linear aeroelastic model stability solution: (a) $\hat{K}_{yDOFeq} = K_\beta$
 (b) $\hat{K}_{yDOFeq} = 0 \cdot K_\beta$.



(a)



(b)

Source: Elaborated by the author.

In Figure 4.1(a), 100% of stiffness in β is considered ($\hat{K}_{yDOFeq} = K_\beta$) and there is no viscous damping applied. For this case, it was observed that there exists one flutter

point, where $V_f = 47.07 \text{ m/s}$ and $f_f = 5.61 \text{ Hz}$, highlighted by the black line.

In Figure 4.1(b), the linear equivalent stiffness coefficient in β is considered as 0% of K_β ($\hat{K}_{yDOF_{eq}} = 0 \cdot K_\beta$). Without the stiffness contribution, the control surface rotation mode frequency decreases and affects the pitch and plunge modes, emerging new flutter mechanisms.

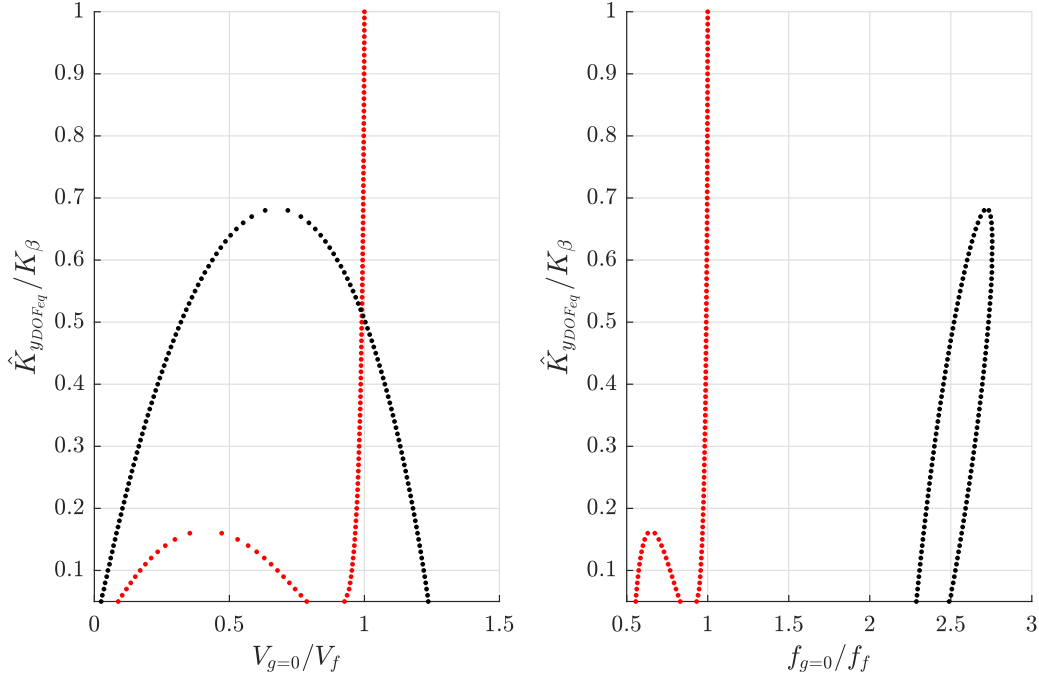
For the linear stiffness variation in $y_{DOF} = \beta$, being

$$0 < \hat{K}_{yDOF_{eq}} < K_\beta$$

Figure 4.2 shows a stability boundary with all unstable points ($V_{g=0}, f_{g=0}$) mapped. The stability boundary shows three flutter mechanisms related to the linear variation, with three ranges of frequencies associated.

The lowest-frequency mechanism and the highest frequency mechanism are related to the change in the torsional stiffness in β . The second lowest frequency mechanism corresponds to the system linear flutter, since the variation of dynamic modes and properties does not change the bifurcation speed.

Figure 4.2 – Stability Boundary for linear control surface stiffness variation.



Source: Elaborated by the author.

4.2 STATE SPACE AEROELASTIC MODEL

Before the process of obtaining the LCO maps in the analyses in the frequency domain and time domain, some verification steps were carried out with the aim of validating the aerodynamic approximation made by the RFA method and the linear dynamics of the system represented by the matrix $\mathbf{A}_{ae}$.

4.2.1 Aerodynamic Approximation by Rational Functions

For the aerodynamic approximation to expand the AICs matrix from frequency domain to Laplace domain, the parameters of reduced frequency and aerodynamic lags are shown in Table 2 and Table 3.

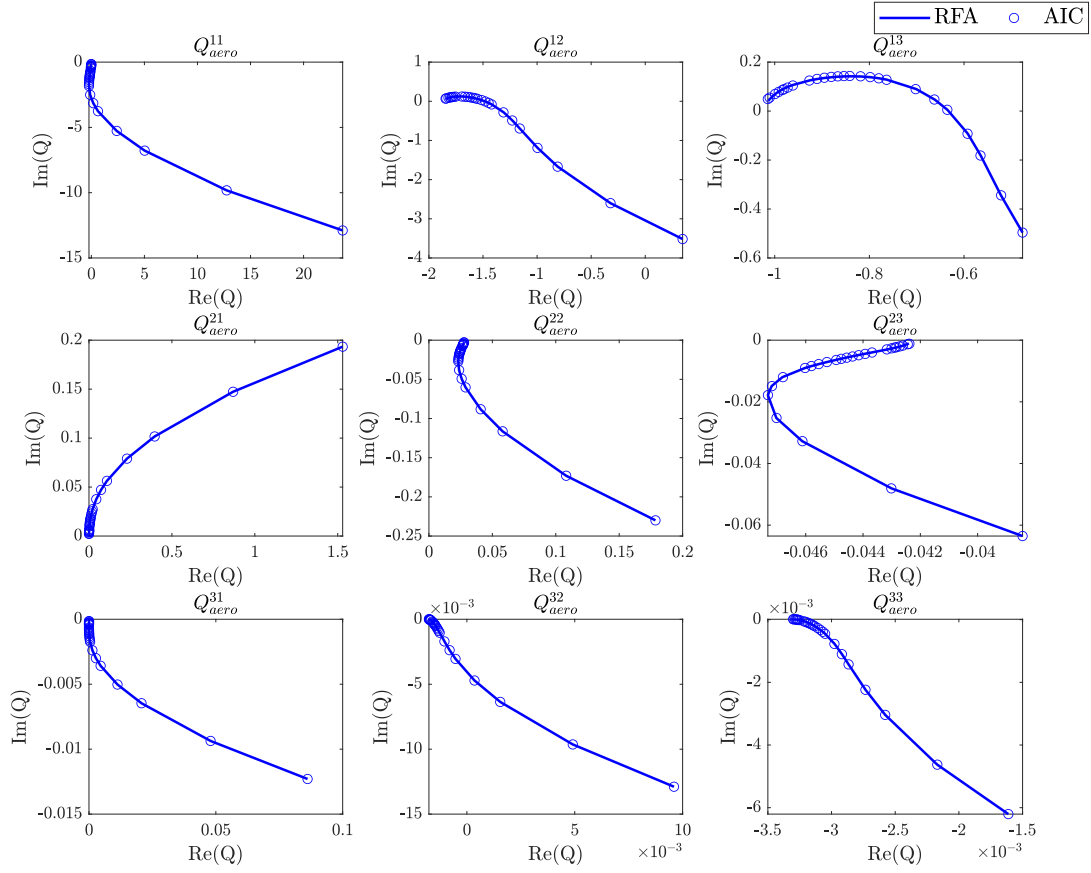
Table 2 – Reduced frequency list applied in the RFA process.

Reduced frequency list			
$k_1 = 0.012$	$k_8 = 0.04$	$k_{15} = 0.12$	$k_{22} = 0.50$
$k_2 = 0.014$	$k_9 = 0.06$	$k_{16} = 0.14$	$k_{23} = 0.75$
$k_3 = 0.020$	$k_{10} = 0.07$	$k_{17} = 0.16$	$k_{24} = 1.00$
$k_4 = 0.024$	$k_{11} = 0.08$	$k_{18} = 0.18$	$k_{25} = 1.50$
$k_5 = 0.028$	$k_{12} = 0.09$	$k_{19} = 0.20$	$k_{26} = 2.00$
$k_6 = 0.030$	$k_{13} = 0.10$	$k_{20} = 0.30$	—
$k_7 = 0.034$	$k_{14} = 0.11$	$k_{21} = 0.40$	—

Table 3 – Lag poles applied in the RFA process.

aerodynamic lag
$p_1 = 0.05$
$p_2 = 0.21$
$p_3 = 0.48$
$p_4 = 0.85$
$p_5 = 1.33$
$p_6 = 1.91$
$p_7 = 2.60$

Figure 4.3 presents the results of RFA application for the aerodynamic lags and reduced frequency parameters. Its possible to see that, for the lag poles selected, the RFA presents good results compared with the AICs matrix.

Figure 4.3 – Comparison between $Q(ik)$ and $Q(s)$ 

Source: Elaborated by the author.

4.2.2 State Space Aeroelastic Matrix

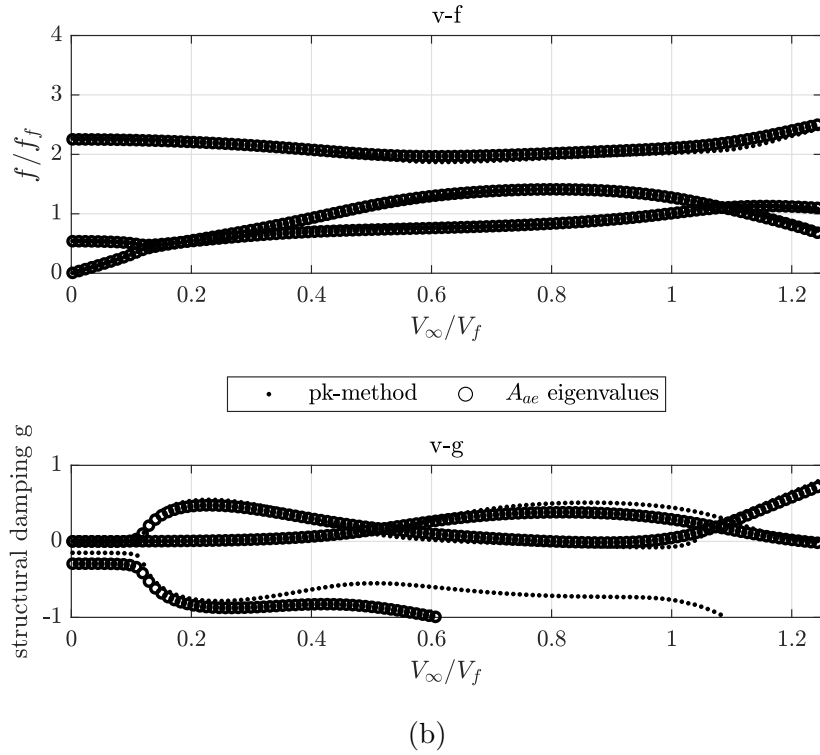
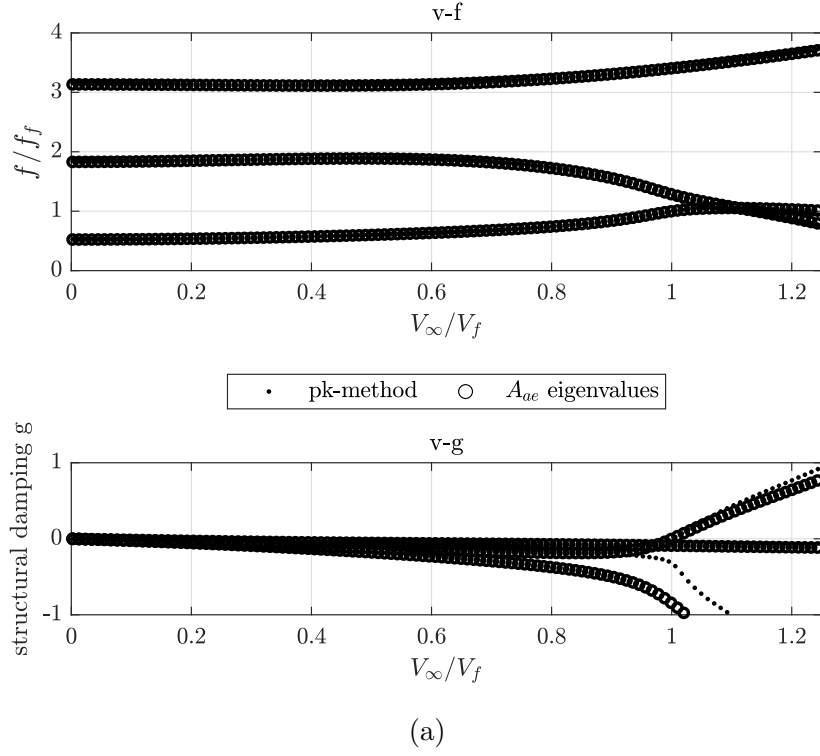
As mentioned in Section 3, the $\mathbf{A}_{ae}$ will be used in the development process of LCO maps in both frequency and time domains. For frequency domain, the matrix will represent the linear equivalent aeroelastic model with different values of linear equivalent stiffness and viscous damping for the control surface DOF in $\mathbf{K}$ and $\mathbf{B}$, considering each condition determined by the interactive process of section 3.6.

For time domain, the matrix will compose the nonlinear model, where the $\mathbf{K}$ is considered a $N_{DOF} \times N_{DOF}$ matrix where $K_\beta = 0$ and $\mathbf{D}$ matrix is considered null $N_{DOF} \times N_{DOF}$ matrix. The stiffness and damping related to the control surface DOF are inserted by nonlinear functions in the time simulation process explained in Section 3.7.

To verify the $\mathbf{A}_{ae}$ composed by structural and aerodynamic portions, its eigenvalue solution was compared with the pk-method results to verify the elastic modes behavior and flutter properties. Figure 4.4 shows the results compare the Flutter solution between

the traditional pk-method and the $\mathbf{A}_{ae}$ eigenvalue extraction.

Figure 4.4 – comparison between v-g-f graph of $\mathbf{A}_{ae}$ eigenvalues extraction and pk-method (a) $\hat{K}_{yDOF} = K_\beta$ (b) $\hat{K}_{yDOF} = 0 \cdot K_\beta$.

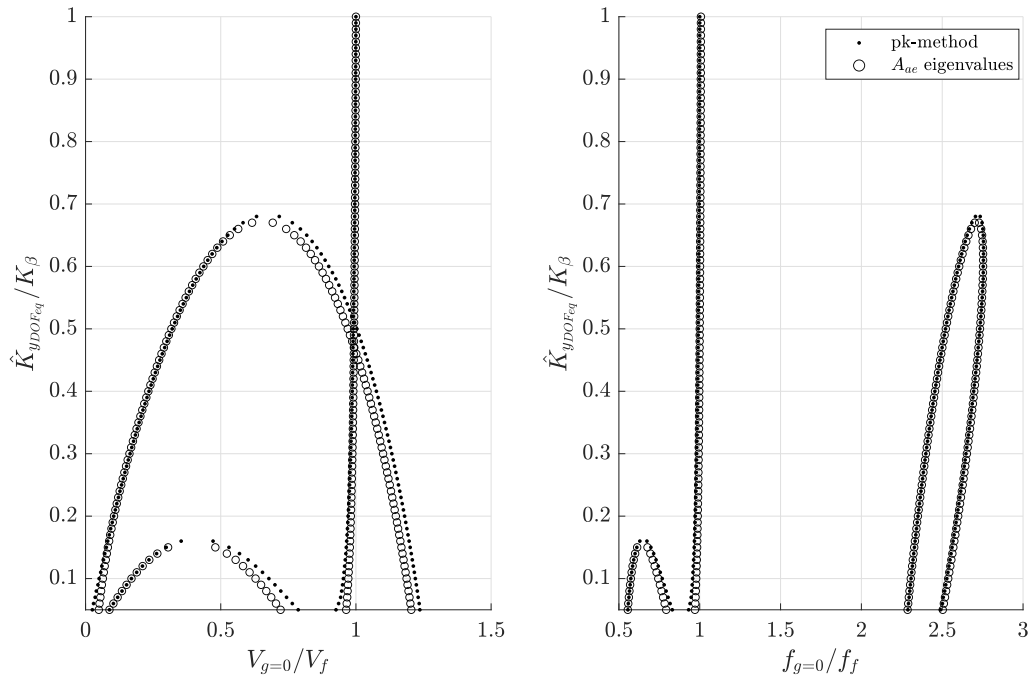


Source: Elaborated by the author.

As the aerodynamic for $\mathbf{A}_{ae}$ is expanded to a complex plan, there are expected differences between the damping. However, it is important that the comparison is similar at the cross of the imaginary axis (flutter points), since the LCO characterization in the frequency domain depends on the parameters $V_{g=0}$ and $f_{g=0}$.

To complete the verification step, Figure 4.5 shows the results of the stability boundary composed by the $\mathbf{A}_{ae}$ eigenvalue problem and its comparison with Figure 4.2, completing the cross-checking analysis.

Figure 4.5 – Comparison between pk-method stability boundary and $\mathbf{A}_{ae}$ stability boundary.



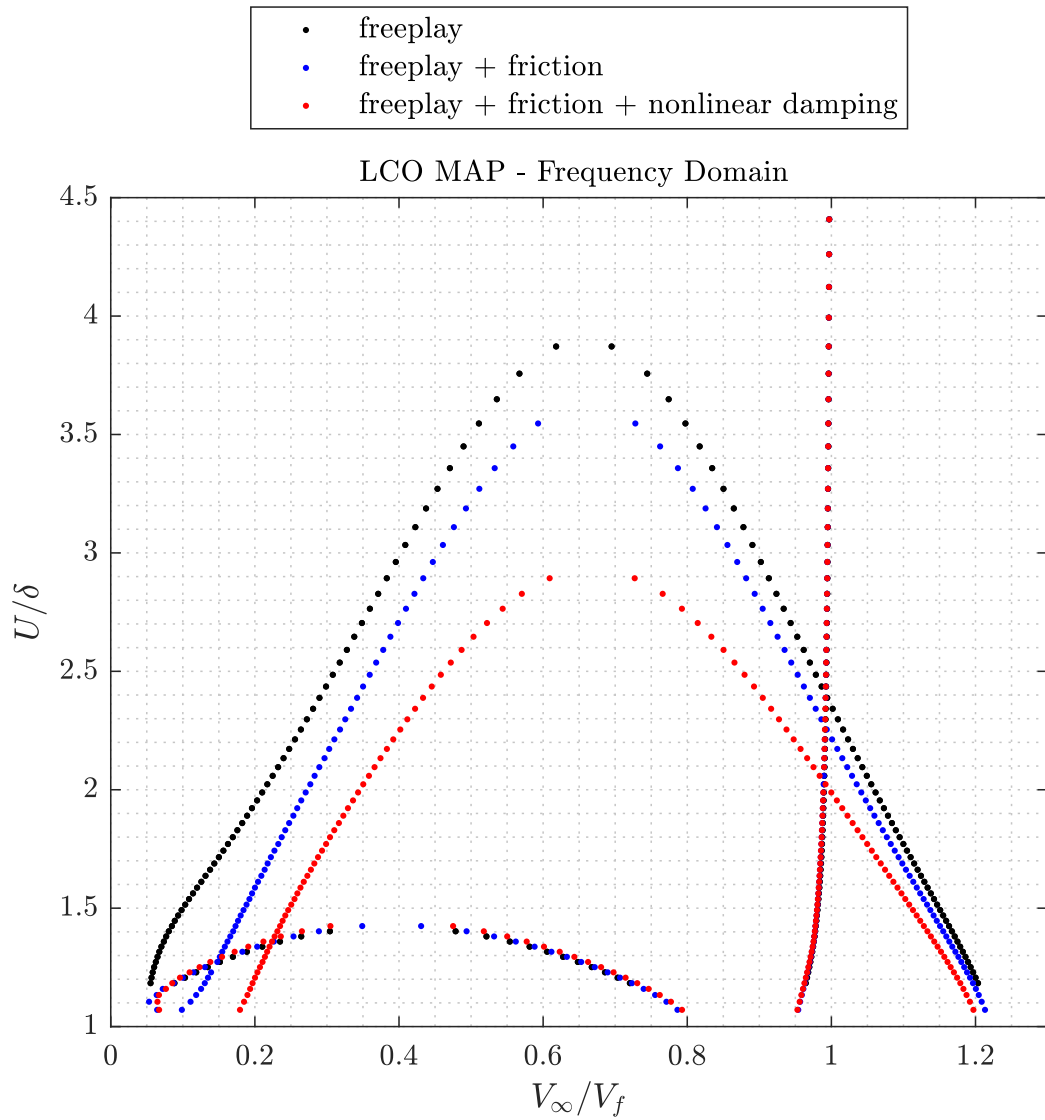
Source: Elaborated by the author.

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4.3 LCO MAP - FREQUENCY DOMAIN

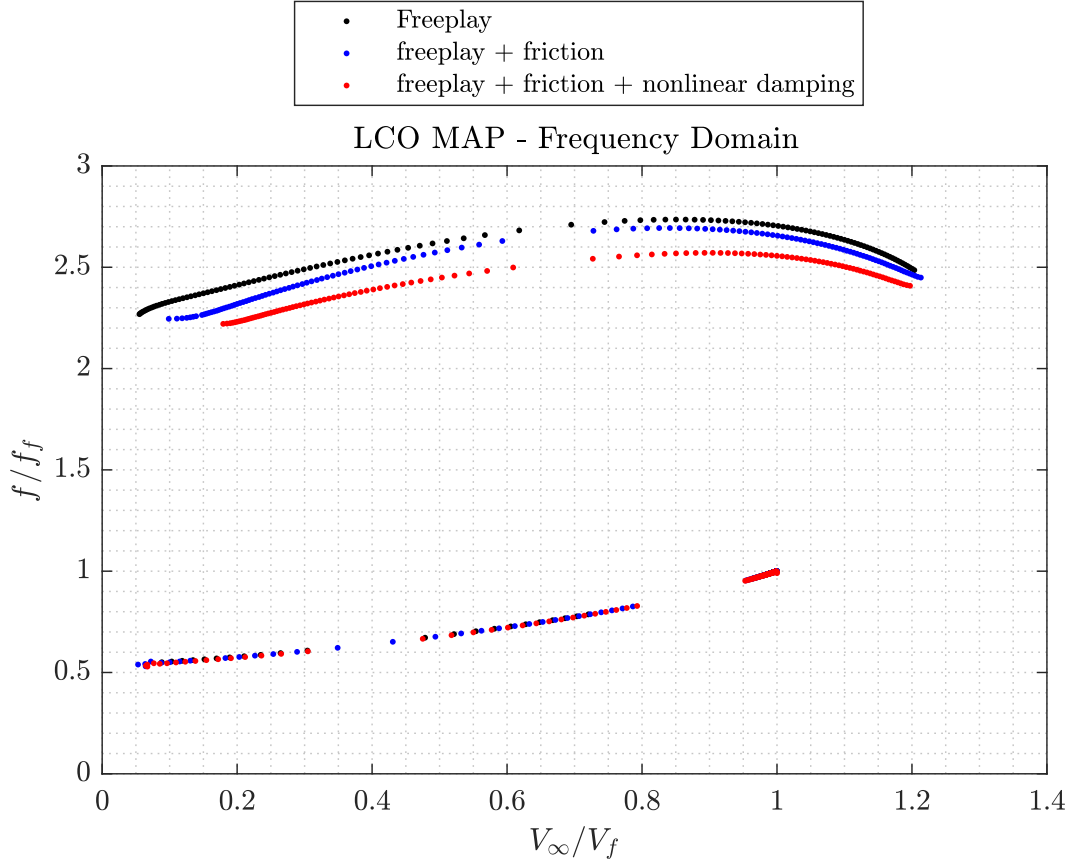
The ELT process with the describing functions generated by HM was applied for 3 cases: 1) just control surface freeplay; 2) control surface freeplay with friction acting simultaneously; and 3) control surface freeplay with friction and nonlinear damping acting simultaneously. The results of LCO amplitude and frequency are shown in Figure 4.6 and Figure 4.7.

Figure 4.6 – LCO frequency map - ELT routine application.



Source: Elaborated by the author.

Figure 4.7 – LCO frequency map in frequency domain considering the ELT process.



Source: Elaborated by the author.

Analyzing the results, there are three LCO curves related to the mechanisms frequencies predicted at the stability boundary of 4.5. The adoption of friction and nonlinear damping causes a LCO amplitude and frequency decrease for the mechanism related to control surface rotation mode. As the other mechanisms are not linked directly to control surface DOF property variation, there are no significant changes in its characteristics.

All these results and interpretations are based on nonlinear aeroelasticity theory and the combination of HM and ELT methods. However, the existence of all the LCO cases mapped by the frequency domain analysis must be confirmed by time domain analyses, as well as stability and oscillatory response behavior.

4.4 LCO MAP - TIME DOMAIN

From the LCO map in the frequency domain illustrated by Figure 4.6 and Figure 4.7, the dynamic response in the time was applied for some initial conditions as shown in Table 4, considering a freeplay width of $\delta = 1^\circ$ (2° peak to peak) for the three cases mapped. Some time simulation results will be shown to explain the general system characteristic, while all the LCO and amplitudes obtained will be illustrated at a time domain LCO map.

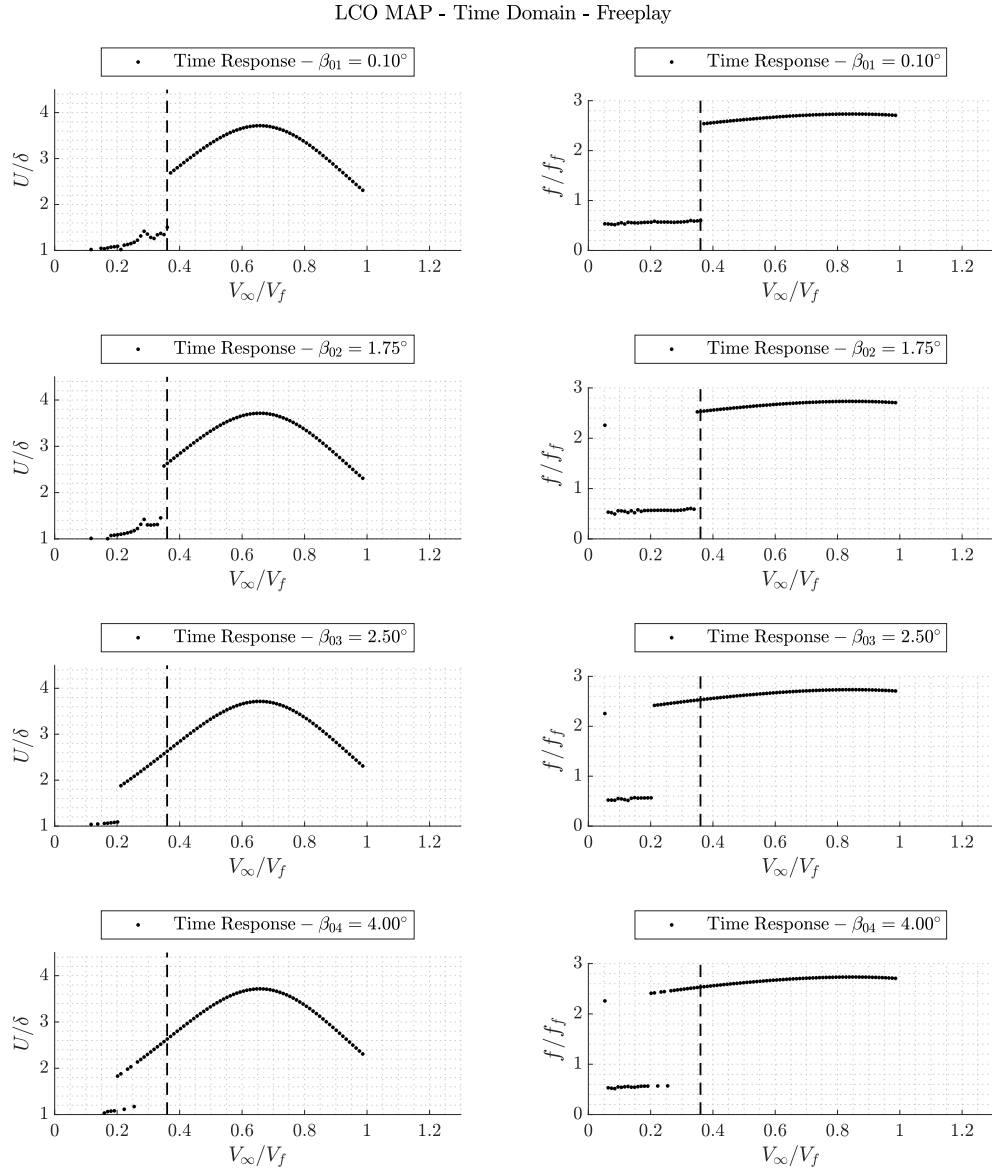
Table 4 – Initial condition of $\beta(t)$ in time simulation

initial conditions of β
$\beta_{01} = 0.10^\circ$
$\beta_{02} = 1.75^\circ$
$\beta_{03} = 2.50^\circ$
$\beta_{04} = 4.00^\circ$

The time domain analysis process was applied to the same three cases predicted by the frequency domain process. Figure 4.8, Figure 4.9 and Figure 4.10 display the time domain results using the state space nonlinear aeroelastic model for the cases of freeplay, freeplay with friction, and freeplay with friction and nonlinear damping, respectively.

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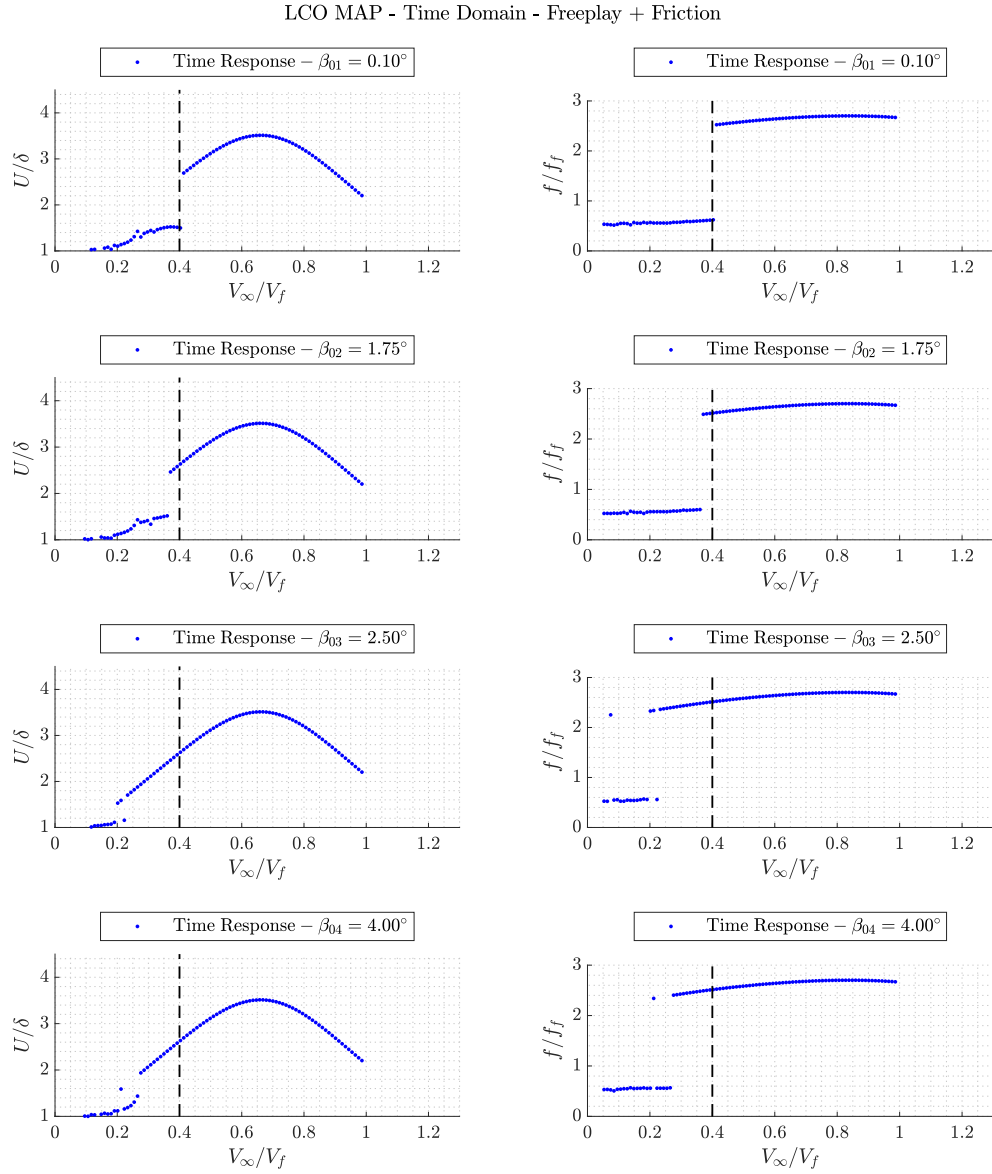
Figure 4.8 – LCO amplitude and frequency for freeplay case.



Source: Elaborated by the author.

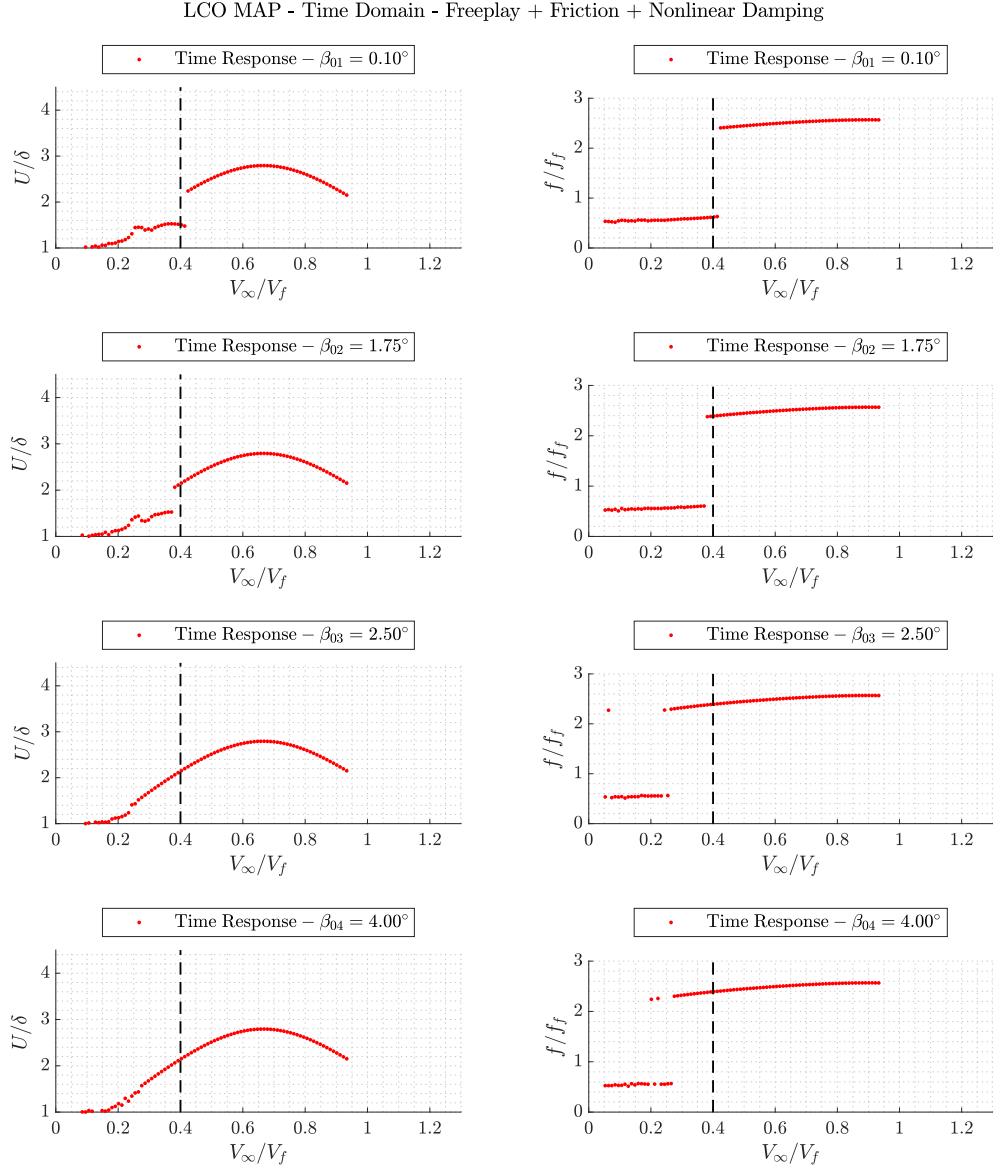
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Figure 4.9 – Results of LCO amplitude and frequency for freeplay with friction case.



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Figure 4.10 – Results of LCO amplitude and frequency for freeplay with friction and nonlinear damping case.

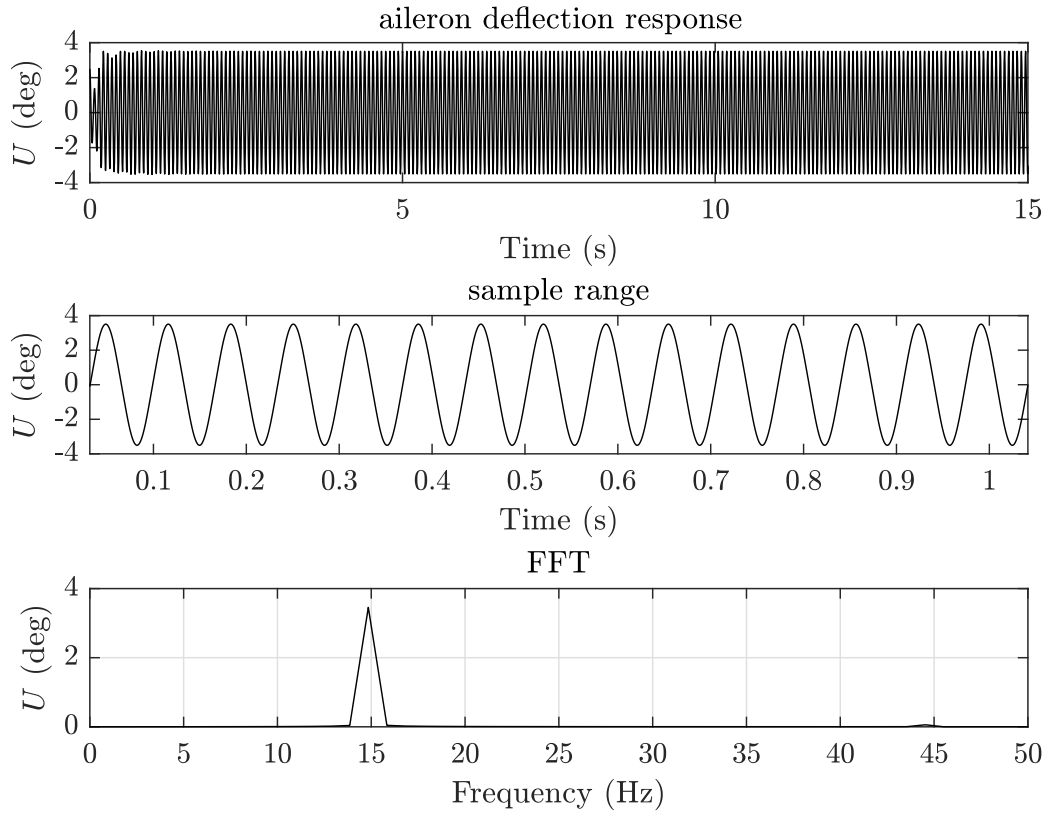


Source: Elaborated by the author.

The amplitude and frequency values are based on the FFT of the response for the same simulation time. For the three cases and after a certain velocity ($0.36 \cdot V_f$ for the first case and $0.40 \cdot V_f$ for the second and third cases) highlighted by dotted lines, the system response always converges to the same LCO in terms of amplitude and frequency, related to the control surface rotation and pitch mode. For these cases, the response behavior is also characterized by a single harmonic motion, as illustrated by Figure 4.11, Figure 4.12,

and Figure 4.13.

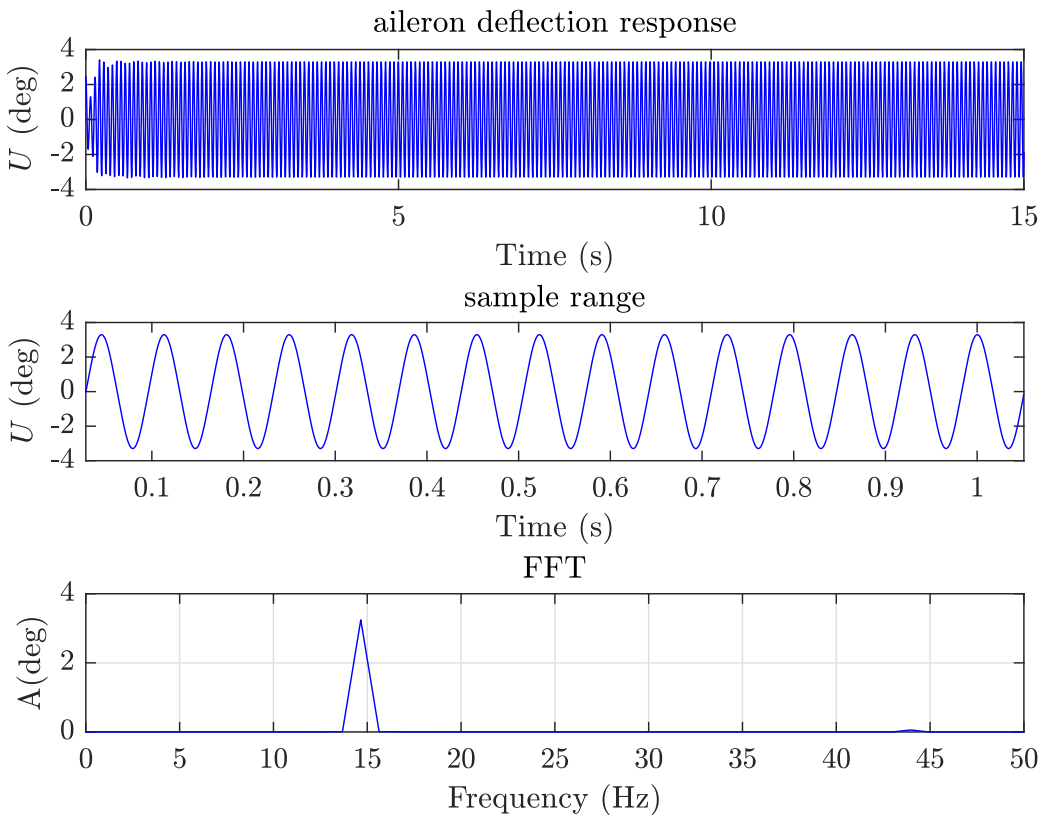
Figure 4.11 – Time response of freeplay for $V_\infty/V_f = 0.53$ and $\beta_0 = 2.5^\circ$



Source: Elaborated by the author.

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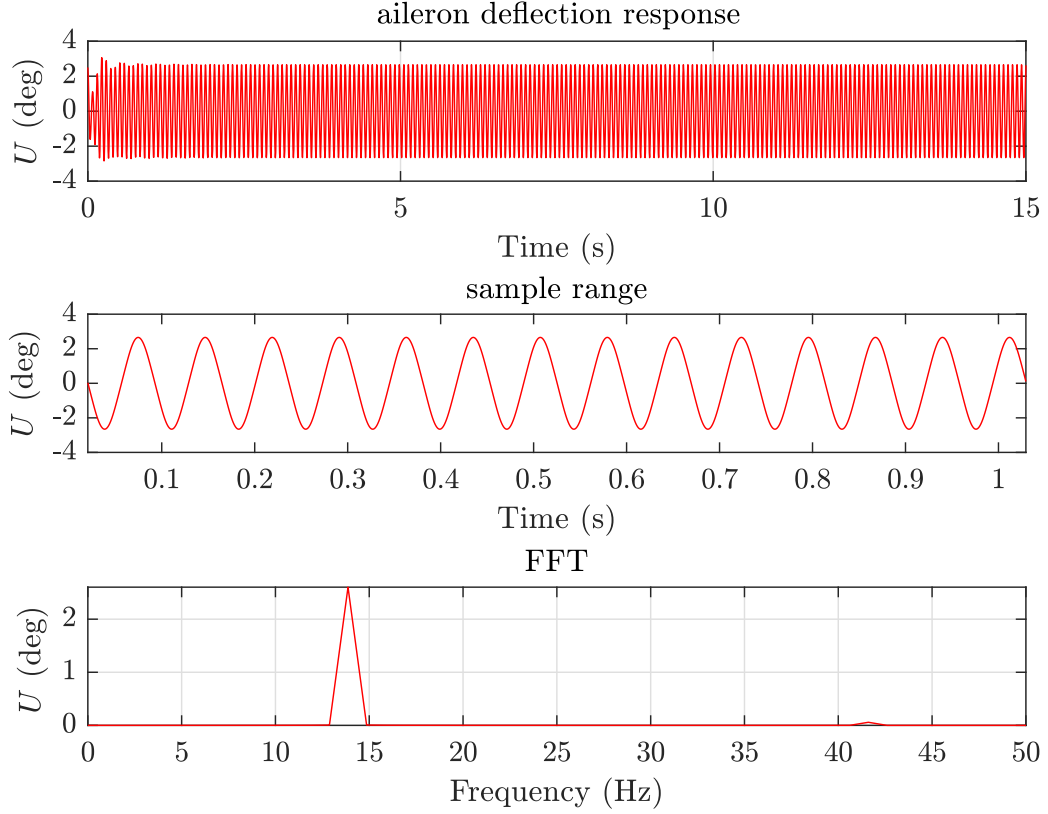
Figure 4.12 – Time response of freeplay with friction for $V_\infty/V_f = 0.53$ and $\beta_0 = 2.5^\circ$



Source: Author.

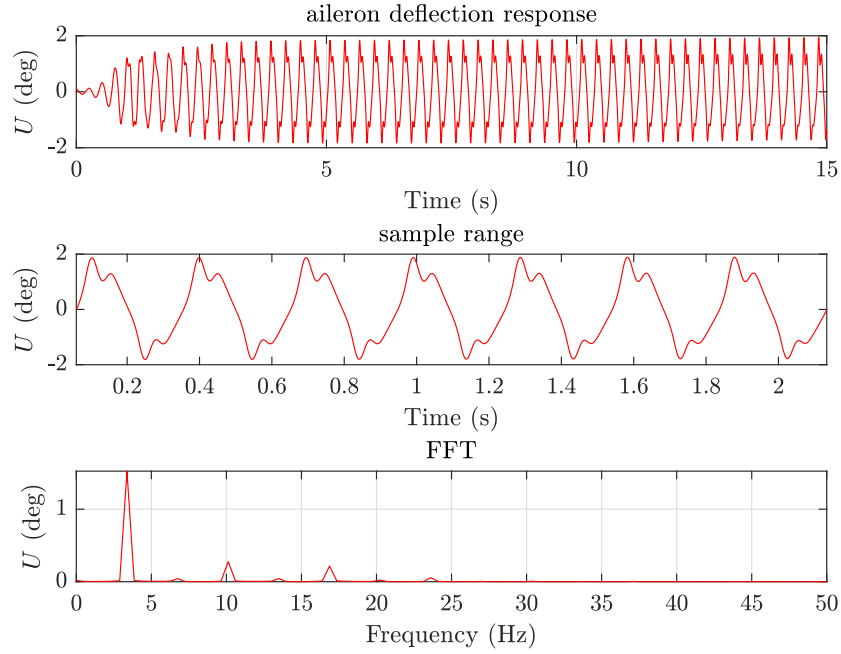
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Figure 4.13 – Time response of freeplay with friction and nonlinear damping case for $V_\infty/V_f = 0.53$ and $\beta_0 = 2.5^\circ$

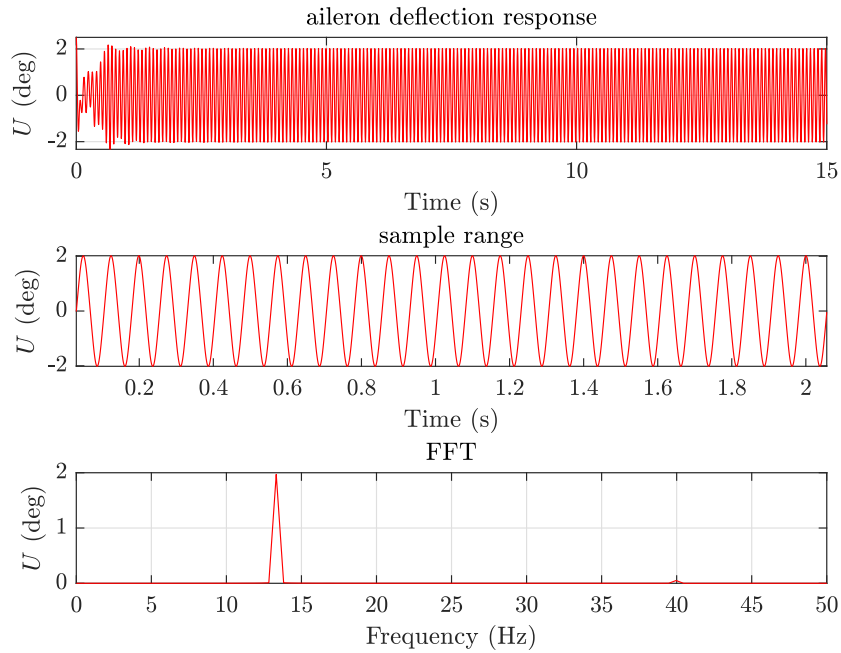


Source: Elaborated by the author.

For the cases before the dotted lines highlighted, the LCO convergence depends on initial conditions, indicating a possible unstable region of the map. An example is illustrated by Figure 4.14, with time responses for the same velocity case $V_\infty/V_f = 0.36$ and different initial conditions. In Figure 4.14(b), the response behavior converges to the LCO related to the highest amplitude and frequency. However, Figure 4.14(a) shows a response behavior represented by more harmonics, with the frequency of the greatest contribution corresponding to the smallest amplitude mechanism, also schematized in the frequency domain LCO map. For the time domain map, these points are represented by the amplitude related to the first harmonic. These mechanisms can also be defined as a stables LCO, but with more harmonics or a bifurcation around these points. This part of system behavior is not detailed in this work.

Figure 4.14 – (a) $\beta_0 = 0.1^\circ$ (b) $\beta_0 = 2.5^\circ$.

(a)



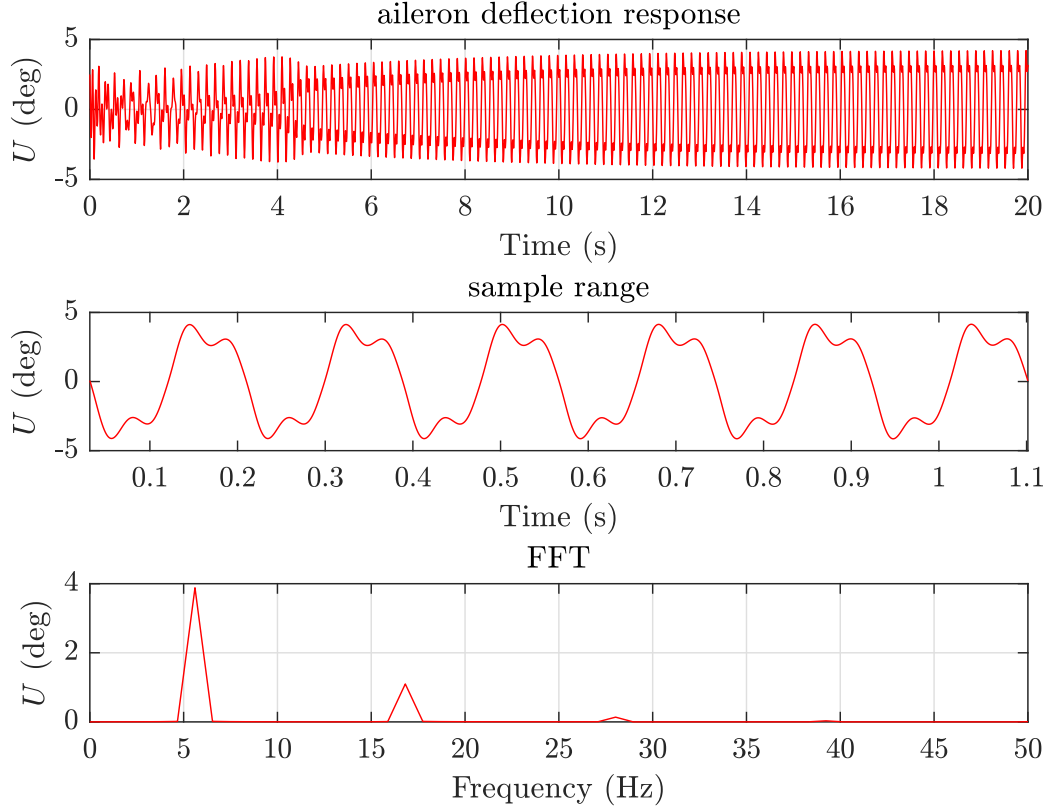
(b)

Source: Elaborated by the author.

For all conditions from $V_\infty/V_f = 0.40$, the LCO is stable until $V_\infty/V_f = 0.98$. For cases close to $V_\infty/V_f = 0.99$, the system enters a new bifurcation region illustrated by Figure 4.15 and change the spectrum to be more harmonic until it becomes unstable for

$$V_\infty/V_f = 1.$$

Figure 4.15 – Time response of freeplay with friction and nonlinear damping case for $V_\infty/V_f = 0.99$ and $\beta_0 = 2.5^\circ$

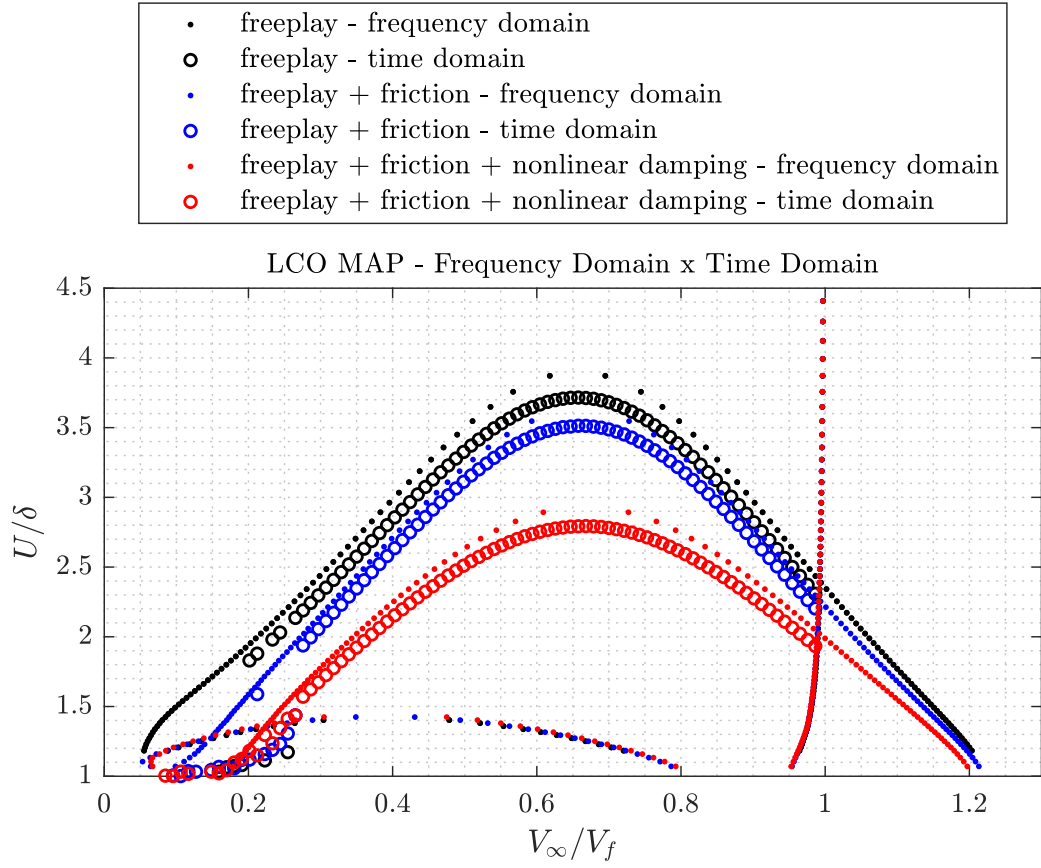


Source: Elaborated by the author.

For cases greater than $V_\infty/V_f = 1$, the system achieves large amplitudes without nonlinearity influence or LCO cases.

Finally, Figures 4.16 and 4.17 displayed the difference between both LCO maps, using the time simulation results of $\beta_0 = 4.00^\circ$. It is possible to observe that the amplitudes and frequencies expected by the frequency domain LCO map are very similar to results obtained by time simulations. Considering all the results shown previously, the region below $V_\infty/V_f = 0.40$ corresponds to a possible unstable LCO region since the convergence of frequency and amplitude changes according to the disturbance of the system. Furthermore, above $V_\infty/V_f = 0.98$, the system becomes unstable, and no LCO is found.

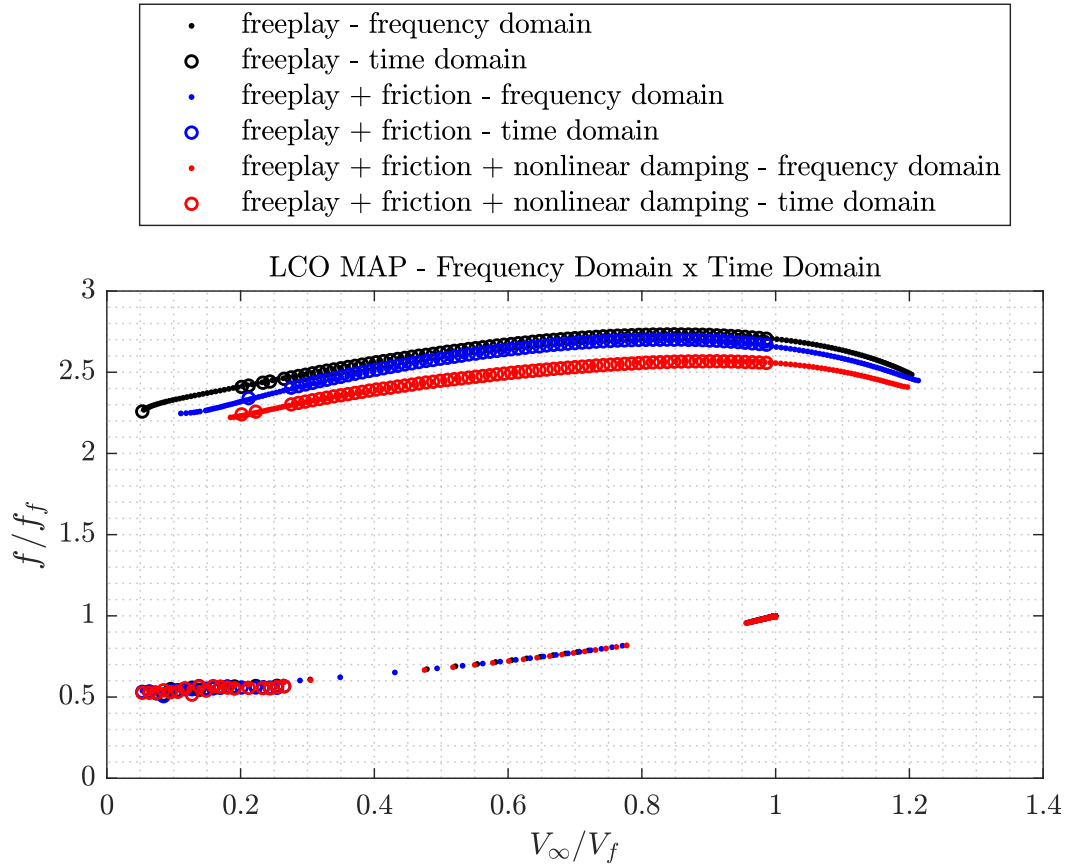
Figure 4.16 – Comparison between amplitudes of frequency domain LCO map and time domain LCO map considering an initial condition of $\beta_0 = 4.00^\circ$.



Source: Elaborated by the author.

(Intentionally left in blank)

Figure 4.17 – Comparison between frequencies of frequency domain LCO map and time domain LCO map considering an initial condition of $\beta_0 = 4.00^\circ$



Source: Elaborated by the author.

5 CONCLUSIONS AND FUTURE WORKS

This work presents a complete process to investigate the effects of structural nonlinearities on an aeroelastic system, applying simple methods and tools to investigate the LCO (Limit cycle oscillation) phenomenon in the time and frequency domains. For this, an aeroelastic physical system that corresponds to a two-dimensional typical section with three DOF, including a trailing edge aileron with freeplay, damper, and friction, was modeled, integrating its inertial, structural, and aerodynamic forces into a single aeroelastic model.

The combination of HM (Harmonic Balance method) and ELT (Equivalent Linearization Technique) allows the investigation of possible LCO cases in the frequency domain through an interactive process that approximates the inclusion of nonlinearities as a function of linear equivalent parameters. The linear equivalent aeroelastic model is represented by a state-space aeroelastic model with the aerodynamic forces approximated to the Laplace domain by RFA (Rational Functions Approximation) method. The linear equivalent system was verified, mainly to compare the flutter point consistency, through a cross-check with the traditional aeroelastic flutter solution, which depends on reduced frequency.

Time simulation tools with numerical methods allow to detail the aileron response over time and analyze the dynamic behavior and stability characteristics of the LCO cases predicted in frequency domain analyses, if successful. For this, a nonlinear aeroelastic model with real nonlinear dynamics was also developed using the state space matrices for the linear part together with a nonlinear Simulink model of freeplay, friction, and damping.

The results presented in the previous section covered all steps of methodology and maintained convergence with the results of this study with other works presented in the literature using the same aeroelastic model. The comparison between maps shows that the approximation of the nonlinear behaviors as simple harmonic motions and the interactive process to match all nonlinearities at the same time assured a good prediction of the LCO amplitude and frequency related to the nonlinear changes in structural characteristics of control surface DOF (Degree-of-freedom), showing and confirmed by time domain analysis that the LCO properties are associated with frequencies and aerolastic modes of structure.

The results also allow the application of the theory present in the literature around the regions of unstable and stable LCO due to different disturbances for the system. In the same region, dynamic system responses can vary from LCO associated with control surface mechanisms to pitch-plunge mechanisms also influenced by control surface, presenting a nonlinear behavior more complex than a single harmonic motion, with more harmonics as contributions. These effects may be due to a bifurcation around that velocity where little disturbances can contribute.

Furthermore, the results show that freeplay, friction, and nonlinear damping on the control surface do not have a strong effect on linear flutter conditions and their amplitude, as the mechanisms involved are related to pitch and plunge coupling. The system, after linear flutter, still becomes exponentially unstable.

For future works, some suggestions are proposed, such as the improvement of processes to be capable of investigating the bifurcation regions, exploring the methodologies around grazing bifurcations, the application of processes in 3D aeroelastic models, and the implementation of other nonlinearities like cubic and bi-linear stiffness, asymmetric freeplay, nonlinear damping, or preloads. Implementations of control laws in the process to evaluate the possibility of avoiding or mitigating the LCO is also interesting.

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