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**COMPARATIVE STUDY OF AERODYNAMIC MODELS
FOR AEROELASTIC FLUTTER PREDICTION AND
DYNAMIC STALL**



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Advisor: Prof. Dr. Higor Luis Silva

Uberlândia - MG

2026

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**ESTUDO COMPARATIVO DE MODELOS
AERODINÂMICOS PARA A PREDIÇÃO DE FLUTTER
AEROELÁSTICO E ESTOL DINÂMICO**

Trabalho de Conclusão de Curso apresentado à
Faculdade de Engenharia Mecânica da Universidade
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Aeronáutica. *VERSÃO REVISADA*

Orientador: Prof. Dr. Higor Luis Silva

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*This work is dedicated to all those who are curious
and encourage education.*

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First, I am deeply grateful to my family, who have always supported and encouraged me in my academic and professional decisions. In particular, I would like to thank my parents, Luciane and Niceio, and my sister, Vitória, who were by my side on a daily basis long before I entered university, offering constant care and motivation. I also extend my sincere thanks to my aunts, uncles, cousins, grandparents, and Teddy, each of whom contributed in their own unique way and holds a special place in my memories.

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*“As invenções são, sobretudo,
o resultado de um trabalho de teimoso.”
(Santos Dumont)*

RESUMO

MEYER MAFFEI, V. J. **Estudo comparativo de modelos aerodinâmicos para a predição de flutter aeroelástico e estol dinâmico**. 2026. 66 p. Trabalho de Conclusão de Curso (Graduação em Engenharia Aeronáutica) – Faculdade de Engenharia Mecânica, Universidade Federal de Uberlândia, Uberlândia - MG, 2026.

Este trabalho revisa e compara os principais métodos para a modelagem do flutter aeroelástico bidimensional, com foco especial nas diferentes formulações aerodinâmicas e na capacidade de representar os efeitos do estol dinâmico. O objetivo é fornecer uma visão comparativa desses métodos, servindo como um conjunto auxiliar de ferramentas para a estimativa e previsão de resultados experimentais, especialmente no contexto de ensaios aeroelásticos em laboratórios de pesquisa.

Inicialmente, são investigadas as formulações mais simples de escoamento colado, o que inclui os modelos quase-estacionário e não estacionário, seguidas pelos modelos semi-empíricos mais avançados para representação do estol dinâmico, nomeadamente as formulações ONERA e Leishman–Beddoes. Todos os modelos são implementados numericamente em MATLAB e analisados por meio da comparação com um caso experimental de referência disponível na literatura.

A partir das principais características, vantagens e limitações de cada abordagem, conclui-se que, entre os métodos baseados em escoamento colado, o modelo aerodinâmico não estacionário de Wagner apresenta previsões mais precisas quando comparado à formulação quase-estacionária, uma vez que o flutter é intrinsecamente um fenômeno dinâmico de alta frequência. Entre os modelos semi-empíricos com representação do estol dinâmico, destaca-se o método de Leishman–Beddoes, que apresentou os resultados mais satisfatórios, principalmente devido à calibração previamente validada dos parâmetros aerodinâmicos.

Palavras-chave: Flutter, Estol dinâmico, Simulação.

ABSTRACT

MEYER MAFFEI, V. J. **COMPARATIVE STUDY OF AERODYNAMIC MODELS FOR AEROELASTIC FLUTTER PREDICTION AND DYNAMIC STALL**. 2026. 66 p. Trabalho de Conclusão de Curso (Graduação em Engenharia Aeronáutica) – Faculdade de Engenharia Mecânica, Universidade Federal de Uberlândia, Uberlândia - MG, 2026.

The present work reviews and compares the main methods for modeling bidimensional aeroelastic flutter, with particular focus on different aerodynamic formulations and their ability to represent dynamic stall effects. The objective is to provide a comparative view of these methods, serving as an auxiliary set of tools for estimating and predicting experimental results, especially in the context of research aeroelastic tests.

Simpler attached-flow formulations, namely quasi-steady and unsteady models, are first investigated, followed by more advanced semi-empirical dynamic stall models, namely the ONERA and Leishman–Beddoes formulations. All models are numerically implemented in MATLAB and analysed through comparison with a reference experimental case from the literature.

By identifying the main characteristics, advantages, and limitations of each approach, it is concluded that, among the attached-flow methods, the unsteady Wagner aerodynamic model provides more accurate predictions than the quasi-steady formulation, since flutter is inherently a high-frequency dynamic phenomenon. Among the semi-empirical models with dynamic stall representation, the Leishman–Beddoes method yields the most satisfactory results, mainly due to the previously validated calibration of its aerodynamic parameters.

Keywords: Flutter, Dynamic stall, Simulation.

LIST OF FIGURES

Figure 1 – Schematic draw of the experimental set-up (left) and side view of the set-up (right). Source: Amandolese, Michelin and Choquel (2011).	18
Figure 2 – Pitch-Plunge problem. Source: Dimitriadis (2017).	21
Figure 3 – Experimental aerodynamic coefficients for a NACA 0012 airfoil in dynamic stall. Source: Dimitriadis (2017).	23
Figure 4 – Conceptual drawing of boundary layer at the leading edge of a static (a) and pitching up (b) airfoil. Source: Dimitriadis (2017).	26
Figure 5 – Different LEV formation mechanisms. Source: Dimitriadis (2017).	27
Figure 6 – Velocity components perceived by a pitching and plunging airfoil. Source: Dimitriadis (2017).	28
Figure 7 – Wagner’s function representation. Source: Author.	31
Figure 8 – Representation of Δc_l and Δc_m . Source: Dimitriadis (2017).	35
Figure 9 – Experimental lift and moment coefficients for a flat plate. Source: Malher (2016).	35
Figure 10 – Lift coefficient (a, b, c) and aerodynamic moment coefficient (d, e, f) of the flat plate as functions of the angle of attack for an amplitude of 36 degrees and three reduced frequencies: 0.01 (a, d), 0.02 (b, e), and 0.03 (c, f). Solid line: represents ONERA model with the parameters from Table 8. Circles: represents experimental measurements. Source: Malher (2016).	37
Figure 11 – Pitch–plunge aeroelastic model. Source: Dimitriadis (2017).	41
Figure 12 – Frequency and real part of the poles for the reference pitch–plunge problem. Source: Amandolese, Michelin and Choquel (2011).	50
Figure 13 – LCO amplitude of the pitching and plunging response versus relative velocity. Source: Amandolese, Michelin and Choquel (2011).	51
Figure 14 – ONERA lift coefficient compared to Malher (2016).	53
Figure 15 – ONERA pitch moment coefficient compared to Malher (MALHER, 2016).	54
Figure 16 – Comparison of natural frequencies (ω_n) and real part of the poles (p_r) with the results of Amandolese, Michelin and Choquel (2011).	57
Figure 17 – ONERA bifurcation compared to Amandolese, Michelin and Choquel (2011).	59
Figure 18 – Leishman–Beddoes bifurcation compared to Amandolese, Michelin and Choquel (2011).	60
Figure 19 – Temporal response obtained with the Leishman–Beddoes model (solid line) compared to Amandolese, Michelin and Choquel (2011) (dashed line) at U_c	61

Figure 20 – Temporal response obtained with the Leishman–Beddoes model (solid line)
 compared to Amandolese, Michelin and Choquel (2011) (dashed line) at

$1.08 U_c$ 61

LIST OF TABLES

Table 1 – Proposed ONERA model parameters for lift and pitch moment. Source: Malher (2016).	36
Table 2 – Description of Leishman–Beddoes aerodynamic states.	38
Table 3 – Structural parameters in SI units used by Amandolese, Michelin and Choquel (2011).	42
Table 4 – Structural parameters in SI units used by Malher (2016).	42
Table 5 – Aerodynamic calibrated coefficients for ONERA model.	53
Table 6 – Adjusted ONERA parameters adopted for aeroelastic simulations.	55
Table 7 – Results obtained for the critical velocity and frequency.	56
Table 8 – Pitch and plunge LCO amplitudes at the critical velocity U_c	58

LIST OF ABBREVIATIONS AND ACRONYMS

DAEP	Département Aérodynamique, Énergétique et Propulsion (Department of Aerodynamics, Energy and Propulsion)
LB	Leishman–Beddoes dynamic stall model
LCO	Limit-cycle oscillation
LE	Leading edge
LEV	Leading-edge vortex
ONERA	Office National d’Études et de Recherches Aérospatiales (French National Office for Aerospace Studies and Research)
RK	Runge–Kutta method
SI	International System of Units
TE	Trailing edge

LIST OF SYMBOLS

a — Non-dimensional distance between the elastic axis and the mid-chord

b — Semi-chord length

c — Chord length of the plate

c_c — Chordwise force coefficient

c_l — Lift coefficient

c_{l_L} — Linear lift coefficient

c_{l_s} — Static lift coefficient

c_m — Pitching moment coefficient

c_{m_l} — Linear moment coefficient

c_{m_s} — Static moment coefficient

c_n — Normal force coefficient

C_h — Plunge damping coefficient

C_α — Pitch damping coefficient

$C(k)$ — Theodorsen function

E_{kin} — Kinetic energy

E_{pot} — Potential energy

g — Wagner input vector

h — Plunge displacement

I_α — Pitch moment of inertia

$I_{n \times m}$ — Identity matrix

k — Reduced frequency

K_h — Plunge stiffness

K_α — Pitch stiffness

l — Lift force

m — Structural mass

m_{xf} — Aerodynamic pitching moment about the elastic axis
 M — Mach number
 p_R — Real part of the eigenvalues
 q — Pitch rate
 $\mathbf{Q}$ — Linear system matrix
 R — Reduced pitch-rate parameter
 S — Static unbalance between pitch and plunge
 t — Time
 U — Freestream airspeed
 U_c — Critical flutter velocity
 U_w — Boundary-layer velocity at the wall
 W_0 — Downwash velocity
 W_1 to W_4 — Wagner model state variables
 x_f — Elastic axis position
 y — State vector
 $0_{n \times m}$ — Zero matrix
 α — Pitch angle / angle of attack
 $\bar{\alpha}$ — Effective angle of attack (Leishman–Beddoes model)
 $\Delta\alpha$ — Pitch sinusoidal amplitude
 Δx_{cg} — Distance between the elastic axis and the center of gravity
 Γ_1 to Γ_3 — ONERA circulation state variables
 $\varepsilon_1, \varepsilon_2$ — Wagner exponential decay constants
 Φ — Wagner function value
 κ_l, κ_m — ONERA nonlinear gain parameters
 λ — Eigenvalues
 λ_l, λ_m — ONERA exponential coefficients
 ρ — Air density
 σ_l, σ_m — ONERA stall delay parameters
 τ — Time constant
 ω_h — Natural frequency of the plunge mode

ω_α — Natural frequency of the pitch mode

ξ — Damping ratio

CONTENTS

1	INTRODUCTION	17
1.1	Context and Motivation	18
1.2	Objectives	19
1.3	Structure of the Work	19
2	LITERATURE REVIEW	20
2.1	Structural model	21
2.2	Stall	22
2.2.1	<i>Static Stall</i>	24
2.2.2	<i>Dynamic Stall</i>	24
2.2.3	<i>Stall Flutter</i>	28
2.3	Aerodynamic Models	29
2.3.1	<i>Quasi-steady</i>	29
2.3.2	<i>Unsteady</i>	30
2.3.2.1	<i>Theodorsen's model</i>	30
2.3.2.2	<i>Wagner's model</i>	31
2.3.3	<i>ONERA Model</i>	33
2.3.4	<i>Leishman–Beddoes model</i>	37
2.4	Numerical Solution	39
3	METHODOLOGY	41
3.1	Study Case	41
3.2	Models Adaptations	43
3.2.1	<i>Quasi-steady Adaptation</i>	43
3.2.2	<i>Unsteady Adaptation</i>	44
3.2.3	<i>ONERA Model Adaptation</i>	45
3.2.4	<i>Leishman–Beddoes Model Adaptation</i>	48
3.3	Validation Cases	49
4	RESULTS AND DISCUSSION	52
4.1	Aerodynamic Tests	52
4.1.1	<i>ONERA model</i>	52
4.2	Aeroelastic Tests	56
4.2.1	<i>Critical Velocity</i>	56

4.2.2	<i>Modes and Frequencies</i>	57
4.2.3	<i>Amplitudes of the LCO</i>	58
4.2.4	<i>Temporal Simulation</i>	60
4.3	Overall Discussion	62
5	CONCLUSIONS	63
5.1	Conclusion	63
5.2	Future Works	65
	BIBLIOGRAPHY	66

INTRODUCTION

IN engineering projects nowadays, several physical domains must be simultaneously considered to ensure safe and efficient designs. In the aeronautical field, one subject that plays a fundamental role is aeroelasticity, as it studies phenomena arising from the interaction between aerodynamic, inertial, and elastic forces acting on flexible structures, as described by Wright and Cooper (2008).

Among the different aeroelastic instabilities, flutter is of particular importance, as it involves the coupling of structural vibration modes in such a way that energy is continuously extracted from the airflow. If the critical flutter speed is exceeded, this instability may lead to catastrophic structural failure.

Under certain assumptions, flutter can be modeled as a linear phenomenon, typically considering small angles of attack, attached flow, and steady aerodynamics. However, when these assumptions are no longer valid, nonlinear effects become significant. Between the nonlinearities, one of the most relevant aerodynamic phenomena is the stall, that occurs due to flow separation at high angles of attack and strong adverse pressure gradients. Once reached, the stall condition is reversible if the angle of attack decreases, enabling the reattachment of the flow.

If an oscillation is observed leading to periodic flow separation and reattachment, then the phenomena is specifically named dynamic stall, including some particularities. To represent such nonlinear aerodynamic behavior, several semi-empirical models have been developed based on experimental observations. Among the most widely used are the ONERA and Leishman–Beddoes models, as described by Dimitriadis (2017). These models allow the inclusion of dynamic stall effects in aeroelastic analyses while maintaining reasonable computational cost.

In this scenario, the present work focuses in representing and identifying this phenomenon by implementation of these semi-empirical aeroelastic models in MATLAB, aiming to predict flutter onset and post-flutter responses, such as limit cycle oscillations.

1.1 Context and Motivation

The motivation for this work is directly connected to an experimental research initiative developed at ISAE-SUPAERO, more specifically within the DAEP (Département Aérodynamique, Énergétique et Propulsion). This initiative aims at the construction of a low-speed wind tunnel experimental setup dedicated to the study of aeroelastic phenomena, particularly flutter. The setup is based on a classical two-degree-of-freedom configuration, in which an airfoil is free to oscillate in plunge and pitch, restrained by springs and damping elements.

The experimental configuration is inspired by previously published low-speed flutter experiments, such as those presented in Amandolese, Michelin and Choquel (2011), and features relatively small geometric dimensions suitable for wind tunnel testing. A schematic representation of the experimental setup is shown in Figure 1.

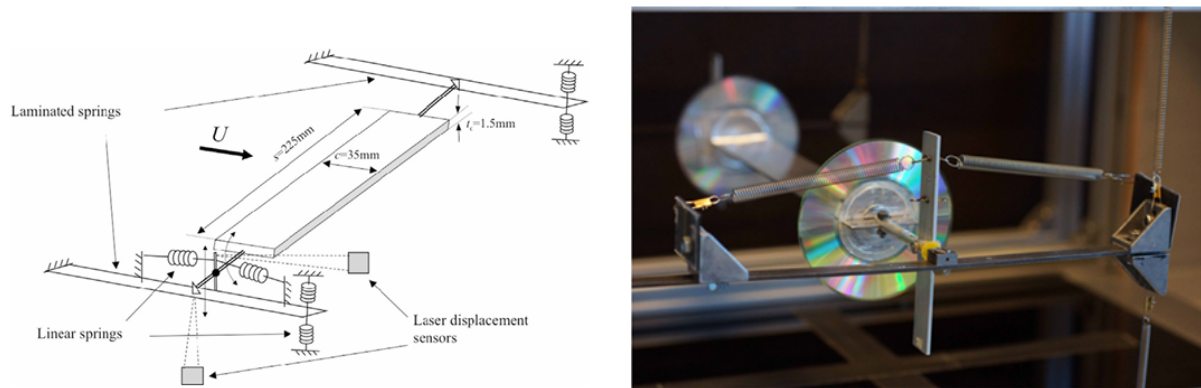


Figure 1 – Schematic draw of the experimental set-up (left) and side view of the set-up (right).
Source: Amandolese, Michelin and Choquel (2011).

Given the pioneering nature of this experimental facility at ISAE-SUPAERO, a parallel numerical effort was proposed to provide theoretical predictions of the system response. The objective is to develop numerical tools capable of estimating the aeroelastic behavior of the setup, thus supporting experiment design, interpretation of results, and validation of aerodynamic models.

In this sense, the present work aims to create an organized numerical tool, incorporating different aeroelastic solution approaches. Starting from classical linear aeroelastic formulations, the study advances toward more complex nonlinear models capable of representing dynamic stall effects. Each modeling approach presents distinct levels of accuracy, limitations, and computational cost, offering a comprehensive set of tools for both design and analysis purposes.

Additionally, the availability of reliable numerical predictions enables a selection and adjustment of key structural and geometric parameters of the experimental setup, such as airfoil chord length, spring stiffness, and damping coefficients.

1.2 Objectives

The main objective of this work is to implement, compare, and evaluate different numerical aerodynamic models for the prediction of aeroelastic flutter onset and post-flutter responses, with particular emphasis on assessing the influence of dynamic stall modeling and the capabilities of each formulation.

Specifically, this work aims to implement linear and nonlinear aerodynamic models in the time domain, incorporate semi-empirical dynamic stall formulations such as the ONERA and Leishman–Beddoes models, compare numerical predictions with available experimental data from the literature, and provide a reliable numerical reference framework to support future aeroelastic experiments conducted at ISAE-SUPAERO.

1.3 Structure of the Work

This work is organized into chapters, each addressing a specific aspect of the study. Chapter 2 presents the literature review, introducing the main sources of nonlinearities in aeroelastic problems, describing the structural pitch–plunge model, and discussing the fundamental characteristics of dynamic stall as well as the aerodynamic models adopted. The models include a range from classical linear formulations to more complex nonlinear representations.

Chapter 3 details the methodology employed in this research, including the definition of the studied problem, the main assumptions adopted, the numerical implementation of the reference aerodynamic models, the time-integration strategy, and the procedures used for model validation.

Chapter 4 presents and discusses the results obtained with each implemented method, providing a comparative and critical analysis that addresses flutter onset, critical speeds, oscillation frequencies, response amplitudes, and hysteresis effects.

Finally, Chapter 5 concludes the work, summarizing the main contributions and highlighting the key insights gained from the numerical investigations.

LITERATURE REVIEW

THIS chapter presents a literature review about the basis of aeroelastic dynamic modeling, creating a theoretical foundation related to the different types and levels of complexity of aerodynamic modeling. It also presents different models associated with structural nonlinearities related to trigonometrical decomposition.

Aeroelasticity, according to Wright and Cooper (2008) and Hodges and Pierce (2011), is the study of the interaction between airflow and flexible structures. The simplest approach is to consider a linearized problem, where displacements are small and both the structure and the airflow exhibit linear behavior.

Nonlinearities in such systems may arise from different physical mechanisms. As discussed in Dimitriadis (2017), the most common sources of nonlinear behavior include:

1. Structural effects, especially those associated with geometry and stiffness variation;
2. Aerodynamic effects, which may incorporate separation phenomena and other features;
3. Control law effects, depending on the adopted architecture and feedback laws.

For the purpose of the present work, only structural and aerodynamic nonlinearities are considered, since no active control strategy is implemented. Therefore, the focus is placed on nonlinear aerodynamic modeling associated with stall flutter, a phenomenon of particular relevance for lifting surfaces operating near stall conditions. For the structure, the nonlinear effect related to geometrical decomposition is studied.

The next sections discuss what the literature states about structural and aerodynamic modeling for a classical pitch–plunge problem, as illustrated in Figure 2. In this configuration, the plunge motion is represented by the vertical displacement h , while the pitch motion is described by the angle α .

The plate is assumed to be connected to two independent springs: K_h , associated with the plunge degree of freedom, and K_α , associated with pitch. Each spring may also include an equivalent damping term, denoted by C_h and C_α , respectively. Furthermore, the plate is assumed to have a total mass m and a moment of inertia about the reference point located at x_f , given by I_α .

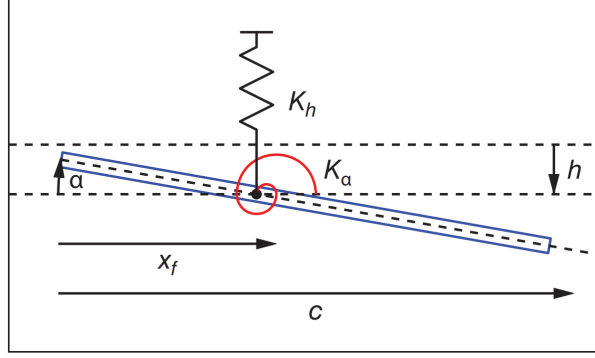


Figure 2 – Pitch-Plunge problem. Source: Dimitriadis (2017).

2.1 Structural model

The structural model considers a plate with two degrees of freedom: the plunge position, h , and the pitch angle, α . The most generic representation of this problem is shown in Eq. (2.1), introduced by Dimitriadis (2017), where the structural matrices are associated with the aerodynamic lift l and moment m_{xf} . It is important to highlight that the aerodynamic term can be calculated in multiple ways, and this term is detailed in the dedicated aerodynamic section.

$$\mathbf{A} \begin{pmatrix} \ddot{h} \\ \ddot{\alpha} \end{pmatrix} + \mathbf{C} \begin{pmatrix} \dot{h} \\ \dot{\alpha} \end{pmatrix} + \mathbf{E} \begin{pmatrix} h \\ \alpha \end{pmatrix} = \begin{pmatrix} -l \\ m_{xf} \end{pmatrix} \quad (2.1)$$

To assign values to the matrices, the simplest approach is to adopt a linear representation for both stiffness and damping, while also assuming small oscillation angles. This assumption, as used by Dimitriadis (2017), leads to the inertia matrix $\mathbf{A}$, damping matrix $\mathbf{C}$, and stiffness matrix $\mathbf{E}$ taking the constant forms shown in Eq. (2.2).

$$\mathbf{A} = \begin{bmatrix} m & S \\ S & I_\alpha \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} C_h & 0 \\ 0 & C_\alpha \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} K_h & 0 \\ 0 & K_\alpha \end{bmatrix} \quad (2.2)$$

In this model, $\mathbf{A}$ is the only matrix with a non-zero secondary diagonal, where the term $S = m \Delta x_{CG}$ represents the unbalance, with Δx_{CG} being the distance between the pivot point and the plate's center of gravity. Because this term appears in the secondary diagonal, it couples the plunge motion with the pitching moment and vice versa, what is also highlighted by Amandolese,

Michelin and Choquel (2011). The special case in which the structural dynamics in pitch are completely decoupled from the plunge occurs when $\Delta x_{CG} = 0$.

For the linearized case, it is possible to obtain the natural frequencies of the pitch and plunge modes by considering small damping. Using the solution of the dynamic analysis, Amandolese, Michelin and Choquel (2011) presents:

$$\omega_h = \sqrt{\frac{K_h}{m}}, \quad \omega_\alpha = \sqrt{\frac{K_\alpha}{I_\alpha}}. \quad (2.3)$$

Although the linear model is the most frequently used in aeroelastic applications, for large amplitudes the initial small-angle trigonometric simplifications become less representative. To address this limitation, Malher (2016) proposes a geometrical decomposition associated with the unbalance parameter S . When writing the kinetic and potential energies, it is obtained:

$$E_{\text{kin}} = \frac{1}{2}M\dot{h}^2 + \frac{1}{2}I_\alpha\dot{\alpha}^2 + S_\alpha\dot{h}\dot{\alpha}\cos(\alpha), \quad (2.4)$$

$$E_{\text{pot}} = \frac{1}{2}K_h h^2 + \frac{1}{2}K_\alpha \alpha^2. \quad (2.5)$$

By applying the Lagrangian formulation to the total energy, the nonlinear dynamic equations are obtained, which also can be written in the form of Eq. (2.1) with the associated matrices shown in Eq. (2.6). We observe that the structural dynamics become more complete, where a cosine term appears in the secondary diagonal terms of the inertia matrix, and the damping matrix includes additional nonlinear coupling terms. The matrices therefore vary as functions of α and $\dot{\alpha}$.

$$\mathbf{A} = \begin{bmatrix} m & S\cos(\alpha) \\ S\cos(\alpha) & I_\alpha \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} C_h & -S\sin(\alpha)\dot{\alpha} \\ -S[\sin(\alpha) + \cos(\alpha)]\dot{\alpha} & C_\alpha \end{bmatrix}, \quad (2.6)$$

$$\mathbf{E} = \begin{bmatrix} K_h & 0 \\ 0 & K_\alpha \end{bmatrix}.$$

Because these matrices depend on the pitch angle and its time derivative, the state-space numerical solution requires time-marching integration. At each time step, the instantaneous values of h , α , $\dot{h}$, and $\dot{\alpha}$ must be used to update the matrices and isolate the accelerations $[\ddot{h} \ \ddot{\alpha}]^T$.

2.2 Stall

Considering an airfoil subjected to an attached airflow, the lift coefficient typically varies linearly with the angle of attack. The term *stall* is used to describe the abrupt, nonlinear loss of

lift that occurs when the angle of attack increases beyond a critical value. This phenomenon is a consequence of the flow separating from the airfoil surface, following the general definitions provided by Anderson John D. (2010).

In this section, the objective is to describe dynamic stall while comparing it with the more commonly known behavior of static stall. Understanding the formulation and characteristics of the dynamic stall phenomenon is essential for interpreting some of the semi-empirical aerodynamic methods presented later. In particular, it is important to recognize how aerodynamic coefficients behave when stall is reached under dynamic conditions. To illustrate the general behavior and the differences between static and dynamic loads, Figure 3 presents experimental data for a NACA 0012 airfoil. The dynamic loads were obtained by McAlister *et al.* (1982) for a case of imposed sinusoidal pitch oscillation between 2° and 22° .

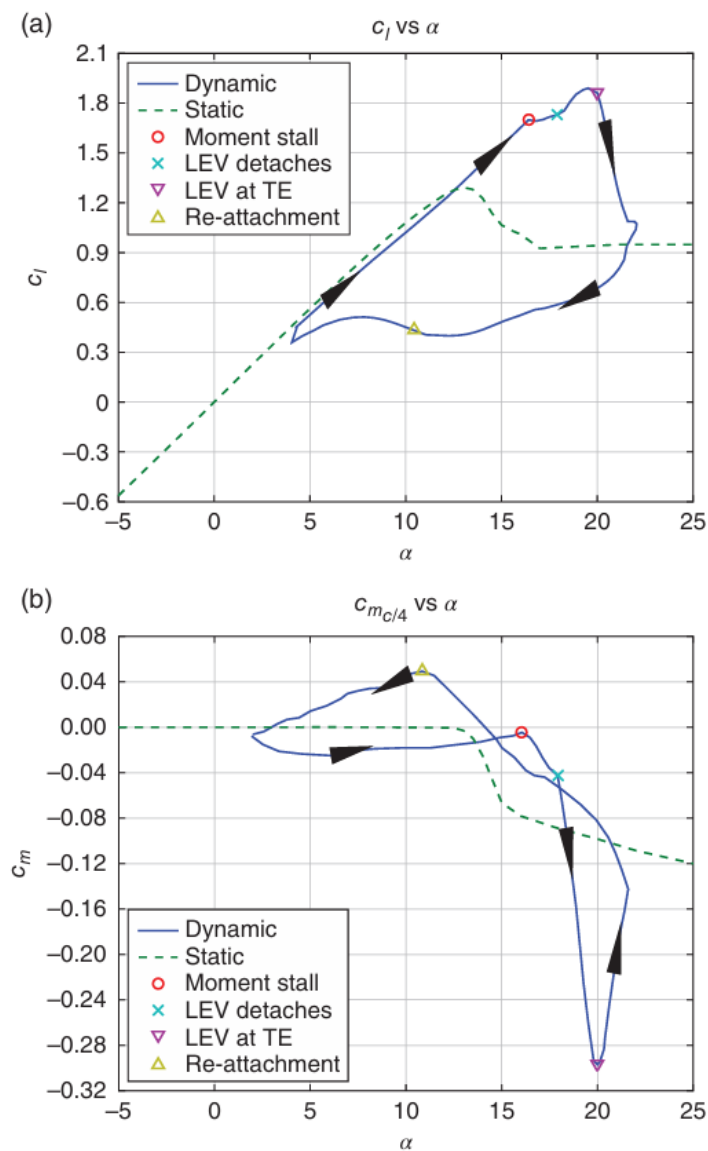


Figure 3 – Experimental aerodynamic coefficients for a NACA 0012 airfoil in dynamic stall. Source: Dimitriadis (2017).

2.2.1 Static Stall

Static stall is the case most commonly treated in the literature. It assumes that the airfoil motion is sufficiently slow such that unsteady effects are negligible and do not significantly influence the aerodynamic loads.

Considering the example shown in Fig. 3 for the NACA 0012 airfoil, the dashed lines represent experimental measurements of the lift and moment coefficients obtained under static conditions. For small angles of attack α , the lift coefficient c_l varies linearly with the angle of attack. As the angle increases, nonlinear behavior begins to appear until the lift reaches a maximum value, which defines the maximum lift coefficient $c_{l,\max}$. Beyond this point, the lift coefficient decreases in a nonlinear manner due to adverse pressure on the outboard surface and the progressive separation of the flow.

Analyzing the moment coefficient about the aerodynamic center c_m , the pre-stall behavior is characterized by a constant value, independent of the angle of attack. As the static stall angle is reached, this constant behavior is no longer observed, and the moment coefficient exhibits negative increments associated with the formation of a separated low-pressure region near the trailing edge. In summary, static stall is characterized by the breakdown of the linear aerodynamic behavior for both lift and moment coefficients.

2.2.2 Dynamic Stall

Dynamic stall is the phenomenon of primary interest in this work. It occurs when the airfoil undergoes motion at considerable high rates such that unsteady aerodynamic effects become significant and influence the aerodynamic loads. Dynamic stall is characterized by the periodic separation and reattachment of the flow around the airfoil, typically occurring at relatively high oscillation frequencies (DIMITRIADIS, 2017).

Referring again to the example shown in Fig. 3, for low angles of attack α , the lift coefficient varies linearly with α , with an initial slope similar to that observed under static conditions. As α increases, the first major difference appears as the linear lift behavior extends beyond the static maximum lift coefficient $c_{l,\max}$. This extension defines the stall delay, which is one of the main characteristics of dynamic stall.

The first nonlinear behavior is typically observed in the moment stall point. At this stage, the lift coefficient begins to deviate from its initial linear gradient due to flow separation on the airfoil upper surface. As α continues to increase, a leading-edge vortex (LEV) forms and subsequently detaches, convecting toward the trailing edge. The presence of the LEV is responsible for a temporary increase in lift, producing a second lift peak. This peak occurs while the LEV remains attached to the airfoil and ends when the vortex reaches the trailing edge.

After this point, the flow becomes fully separated and the lift coefficient decreases abruptly. During the subsequent decrease of α , the lift values may fall below the static reference

until flow reattachment occurs, which itself is delayed and takes place at angles lower than those corresponding to the static maximum lift. Once reattachment is achieved, the airfoil gradually recovers a linear lift behavior and the cycle restarts.

A similar sequence of events can be observed in the pitching moment coefficient. For low angles of attack, c_m remains close to the static value, with small variations caused by angular motion and the resulting non-uniform velocity and incidence distribution over the airfoil. This behavior is disrupted at the moment stall, where initial leading-edge separation generates a nose-down pitching tendency. As the LEV detached and convects downstream, the pitching moment reaches its maximum negative value when the vortex is located near the trailing edge. After the LEV leaves the airfoil, the pitching moment recovers, while the flow remains fully separated. As the angle of attack decreases further, flow reattachment occurs and the moment coefficient gradually returns toward its static behavior.

It is important to note that the aerodynamic response during the increasing and decreasing phases of α differs significantly, demonstrating the strong hysteresis characteristic of dynamic stall. Furthermore, the example presented corresponds to an imposed harmonic motion. Therefore, variations in the aerodynamic response are expected when the airfoil is allowed to move freely or when different oscillation kinematics are considered.

After looking for the expected behavior, it is possible to discuss in more details where these phenomenon come from. Thus, to better understand these particularities, Dimitriadis (2017) classifies the three main mechanisms of the the dynamic stall:

- Stall delay
- Leading Edge Vortex
- Re-attachment

The stall delay is the effect allowing the airfoil to reach lift coefficients higher than the static $c_{l_{max}}$. Two main mechanisms have been identified as responsible for the stall delay effect, one inviscid and one viscous (DIMITRIADIS, 2017). The inviscid mechanism is related to a reduction in the adverse pressure gradient over most of the surface of an airfoil that is pitching up, as compared to the pressure gradient acting on the surface of a static airfoil at the same instantaneous angle of attack. This reduction means that the boundary layer is further from stall during the upstroke and therefore remains attached up to higher angles of attack.

The viscous mechanism for stall delay is referred to as the ‘moving wall’. On a static airfoil the boundary layer velocity at the wall, U_w , is equal to zero since the wall is static, as seen in Figure 4 (a). On an airfoil that is pitching up, the leading edge is moving up in a direction parallel to the boundary layer flow, as shown in Figure 4 (b). As the boundary layer sticks to the surface, its speed is non-zero at the wall. This phenomenon constitutes a significant momentum

input to the boundary layer and therefore delays stall. It only has a significant effect very close to the leading edge. At the downstroke, the moving wall's effect is inverted it forces the flow to separate at the leading edge, thus delaying reattachment.

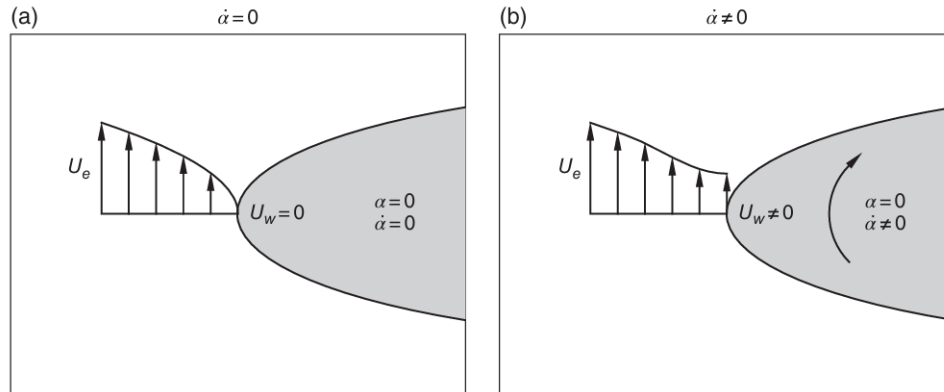


Figure 4 – Conceptual drawing of boundary layer at the leading edge of a static (a) and pitching up (b) airfoil. Source: Dimitriadis (2017).

The second main effect is the Leading Edge Vortex (LEV), mentioned as most striking difference between static and dynamic stall (DIMITRIADIS, 2017). The three main mechanisms responsible for the formation and shedding of the LEV are represented in Figure 5.

At the leading edge laminar separation bubble bursting, the boundary layer near the leading edge is laminar, separating first. The separated shear layer becomes turbulent and reattaches, forming a separation bubble. As the pitch angle increases the bubble bursts abruptly, creating the LEV that then propagates downstream. There could be a certain amount of trailing edge separation, but it does not interfere with the LEV formation process. This mechanism occurs on airfoils with sharp leading edges.

At the mixed leading edge/trailing edge stall, flow separation appears first at the trailing edge and starts to move upstream. As the pitch angle increases, the turbulent boundary layer on the entire front section of the airfoil separates abruptly and the LEV is formed.

At the shock-induced separation case, considering higher Mach numbers, strong shocks form near the leading edge. They impose a very high adverse pressure gradient on the boundary layer, forcing separation. The LEV forms at the foot of the last strong shock, is oblong in shape and is initially enveloped in supersonic external flow.

The LEV causes a considerable suction peak on the upper surface and increases the lift. As it moves past the quarter-chord it also causes a very important nose down pitching moment. Its propagation velocity lies between 35 and 55% of the free stream velocity, depending on the airfoil and the flow conditions. Note that stall delay is independent to LEV formation, although they both cause the maximum lift coefficient to increase far beyond its static value. Secondary vortices can be shed after the passage of the primary LEV under certain circumstances.

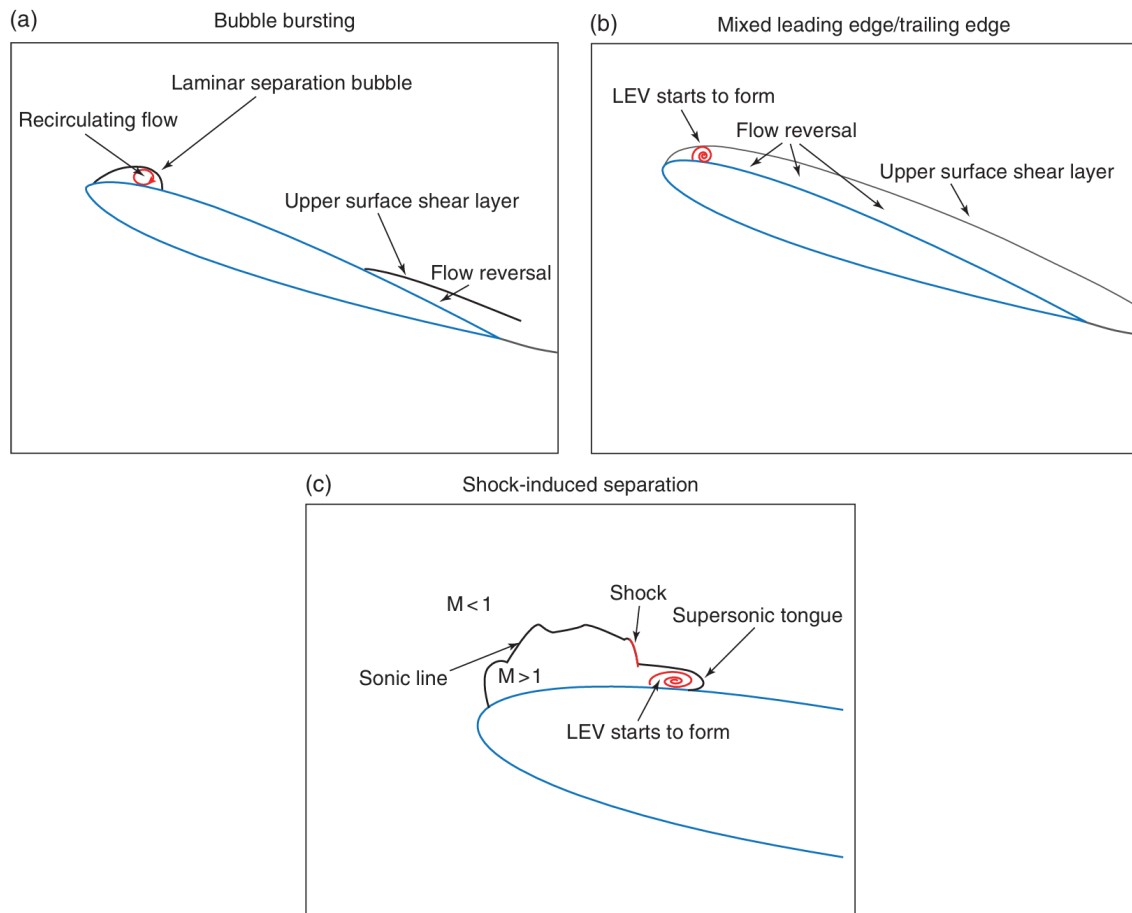


Figure 5 – Different LEV formation mechanisms. Source: Dimitriadis (2017).

The final stage of the dynamic stall process is re-attachment, generally starting at the front of the airfoil and progresses downstream to the trailing edge. The complete dynamic stall process is not entirely repeatable; small cycle-to-cycle variations can be observed and re-attachment is the less repeatable part of the phenomenon. The smooth lift, drag and moment curves published in the literature (e.g. Figure 3) are all obtained by means of ensemble averaging over many cycles (DIMITRIADIS, 2017).

In resume, dynamic stall is very complex phenomena and many parameters can change significantly the flow dynamics, with some of these interdependent. For example, if the oscillation frequency is high enough there can be locally supersonic flow even when the free stream Mach number is low. While different dynamic stall mechanisms have been observed in detail, there is still no full explanation for the occurrence of these mechanisms and no complete theory, so dynamic stall events cannot be predicted from first principles. Measured data has been used by several researchers in order to derive empirical criteria for the occurrence of dynamic stall events and for the prediction of aerodynamic load responses during dynamic stall. The Leishman–Beddoes model and the ONERA model are the most popular and widely used.

2.2.3 Stall Flutter

Stall flutter is generally described as a self-sustained aeroelastic oscillation in which the flow periodically separates from and reattaches to an aerodynamic body (DIMITRIADIS, 2017). A common outcome of stall flutter is the development of a *limit cycle oscillation* (LCO), which distinguishes it from classical linear flutter, where oscillation amplitudes grow without bound once instability is reached.

In contrast to the dynamic stall example shown in Fig. 3, where the pitching motion is externally imposed, stall flutter considers a configuration in which the body is free to move, typically in pitch and plunge. In this case, the aerodynamic loads generated by dynamic stall interact with the structural degrees of freedom, leading to a self-excited oscillatory response. This coupling between unsteady aerodynamics and structural motion is the defining feature of stall flutter.

Modeling stall flutter is particularly challenging due to the strong influence of pitching motion on the aerodynamic response. As discussed by Dimitriadis (2017), pitching induces a chordwise-varying velocity and incidence distribution along the airfoil, effectively modifying the instantaneous geometry perceived by the flow, as represented in Fig. 6. Under attached-flow conditions, this effect can be represented in a quasi-steady. However, once significant flow separation occurs, the quasi-steady assumptions are no longer valid requiring more robust aerodynamic formulations. In essence, the core difficulty in stall-flutter modeling lies in the accurate representation of dynamic stall.

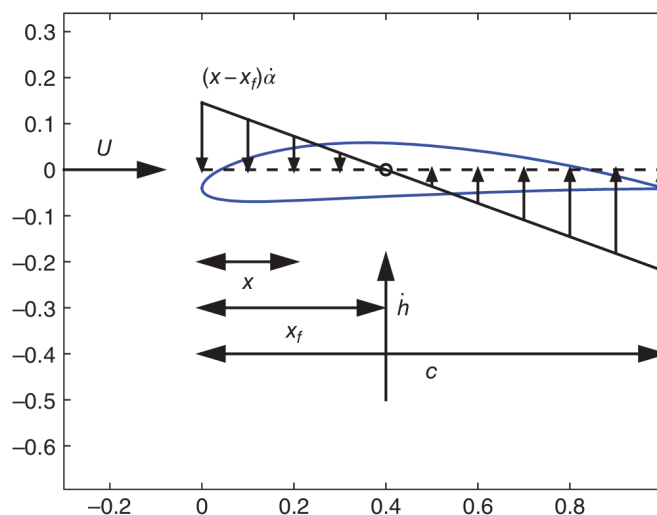


Figure 6 – Velocity components perceived by a pitching and plunging airfoil. Source: Dimitriadis (2017).

Stall flutter is sometimes compared to galloping, but the two phenomena originate from distinct physical mechanisms. Galloping is typically associated with transverse oscillations of bluff bodies under persistent flow separation, whereas stall flutter is primarily linked to the pitch

or pitch–plunge dynamics of streamlined lifting surfaces undergoing cyclic flow separation and reattachment.

2.3 Aerodynamic Models

Depending on the objectives of the analysis and the required level of fidelity, a wide range of aerodynamic models is available in the literature. These models differ in complexity, underlying assumptions, and their ability to represent unsteady and nonlinear flow phenomena.

In this section, the aerodynamic models of interest are presented. First, classical linear models are described, including quasi-steady and unsteady formulations that account for airfoil motion in the calculation of aerodynamic loads but do not explicitly capture stall phenomena. These models are widely used in aeroelastic representations due to their simplicity and computational efficiency and remain suitable for attached-flow conditions.

Subsequently, it is introduced semi-empirical aerodynamic models capable of representing stall-related nonlinearities: the ONERA model and the Leishman–Beddoes model. They are specifically designed to capture unsteady flow separation, hysteresis effects, and nonlinear aerodynamic behavior associated with dynamic stall.

2.3.1 Quasi-steady

The simplest aerodynamic representation commonly adopted in flutter analyses is the quasi-steady assumption. In this approach, the aerodynamic loads are assumed to depend only on the instantaneous motion of the airfoil, neglecting any memory or time-delay effects associated with the flow.

Under the quasi-steady assumption, the lift l and pitching moment m_{xf} appearing in Eq. (2.1) are expressed as linear functions of the position states and their time derivatives (h , α , $\dot{h}$, and $\dot{\alpha}$). These relations are obtained by multiplying the state variables by constant aerodynamic coefficients.

This formulation leads to the coupled aeroelastic system shown in Eq. (2.7), where $\mathbf{y} = [h \ \alpha]^T$. The matrices $\mathbf{A}$, $\mathbf{C}$, and $\mathbf{E}$ represent the structural inertia, damping, and stiffness, respectively, while the matrices $\mathbf{B}$, $\mathbf{D}$, and $\mathbf{F}$ collect the aerodynamic contributions, as presented by Dimitriadis (2017). In this context, the aerodynamic forces are incorporated into the equations of motion in an equivalent mass, damping, and stiffness form.

$$(\mathbf{A} + \rho \mathbf{B})\ddot{\mathbf{y}} + (\mathbf{C} + \rho U \mathbf{D})\dot{\mathbf{y}} + (\mathbf{E} + \rho U^2 \mathbf{F})\mathbf{y} = \mathbf{0} \quad (2.7)$$

While the structural matrices are identical to those introduced in Section 2.1, the equiva-

lent aerodynamic matrices adopted by Dimitriadis (2017) are given in Eq. (2.8).

$$\begin{aligned} B &= b^2 \begin{bmatrix} \pi & -\pi ab \\ -\pi ab & \pi b^2 \left(\frac{1}{8} + a^2\right) \end{bmatrix}, & D_1 &= b^2 \begin{bmatrix} 0 & \pi \\ 0 & \pi \left(\frac{1}{2} - a\right) b \end{bmatrix} \\ D_2 &= \begin{bmatrix} 2\pi b & 2\pi b^2 \left(\frac{1}{2} - a\right) \\ -2\pi b^2 \left(a + \frac{1}{2}\right) & -2\pi b^3 \left(a + \frac{1}{2}\right) \left(\frac{1}{2} - a\right) \end{bmatrix}, & F &= \begin{bmatrix} 0 & 2\pi b \\ 0 & -2\pi b^2 \left(a + \frac{1}{2}\right) \end{bmatrix} \\ & & D &= D_1 + D_2 \end{aligned} \quad (2.8)$$

The geometric parameters a and b are constants related to the airfoil dimensions and reference axis. In particular, b corresponds to the semi-chord length:

$$b = \frac{c}{2}, \quad (2.9)$$

while a denotes the nondimensional distance between the elastic axis and the mid-chord, defined as:

$$a = \frac{x_f}{b}. \quad (2.10)$$

It is worth noting that the quasi-steady formulation can be interpreted as a particular case of unsteady aerodynamic theory, presented in the following section, in which the reduced-frequency effects are neglected. In this sense, the aerodynamic airloads obtained from the unsteady theory given by Eqs. (2.11) and (2.12) reduce to the quasi-steady expressions when the Theodorsen function is set to $C(k) = 1$.

2.3.2 Unsteady

A more refined aerodynamic representation consists in assuming unsteady flow effects, in which the aerodynamic loads depend not only on the instantaneous motion of the airfoil but also on its past motion. In this case, the wake dynamics introduce a memory effect that influences the present aerodynamic response. The main assumptions still include inviscid and attached flow, meaning that flow separation and dynamic stall are not represented. Nevertheless, unsteady aerodynamic models provide improved accuracy over quasi-steady formulations and are widely used for flutter prediction.

2.3.2.1 Theodorsen's model

Theodorsen's model computes the unsteady aerodynamic lift and pitching moment about the elastic axis x_f using Eqs. (2.11) and (2.12), as described by Dimitriadis (2017).

$$l = 2\pi b \rho U C(k) W_0 + \pi \rho b^2 (\ddot{h} - ab\ddot{\alpha} + U\dot{\alpha}) \quad (2.11)$$

$$m_{x_f} = 2\pi b^2 \rho U \left(\frac{1}{2} + a\right) C(k) W_0 - \pi \rho b^2 \left(-ab\ddot{h} + Ub \left(\frac{1}{2} - a\right) \dot{\alpha} - b^2 \left(\frac{1}{8} + a^2\right) \ddot{\alpha}\right) \quad (2.12)$$

The unsteady effects are incorporated through the Theodorsen function $C(k)$, which depends on the reduced frequency k :

$$k = \frac{\omega b}{U}, \quad (2.13)$$

where ω is the oscillation frequency, b is the semi-chord, and U is the freestream velocity. Also, the instantaneous downwash velocity W_0 is defined as:

$$W_0 = U\alpha + \dot{h} + b\left(\frac{1}{2} - a\right)\dot{\alpha}. \quad (2.14)$$

Theodorsen's formulation is particularly useful for flutter analysis in the frequency domain, as it enables the evaluation of system stability and pole migration as the flow velocity varies. However, since the Theodorsen function $C(k)$ is defined for harmonic oscillations, the model is only applicable in frequency-domain based and is not directly suitable for time-domain simulations. Consequently, for time-domain analyses and transient simulations, alternative formulations such as Wagner's model are commonly employed.

2.3.2.2 Wagner's model

Wagner's model is based on Wagner's function $\Phi(t)$, which describes the temporal evolution of a perturbation in the final result. The function starts at $\Phi(0) = 0.5$ and asymptotically approaches unity, as represented in Fig. 7, representing the gradual development of circulation following an instantaneous perturbation. In this formulation, Wagner's function is approximated using a sum of exponential terms with constants $\Psi_1 = 0.165$, $\Psi_2 = 0.335$, $\varepsilon_1 = 0.0455$, and $\varepsilon_2 = 0.3$, giving:

$$\Phi(t) = 1 - \Psi_1 e^{-\varepsilon_1 U t / b} - \Psi_2 e^{-\varepsilon_2 U t / b} \quad (2.15)$$

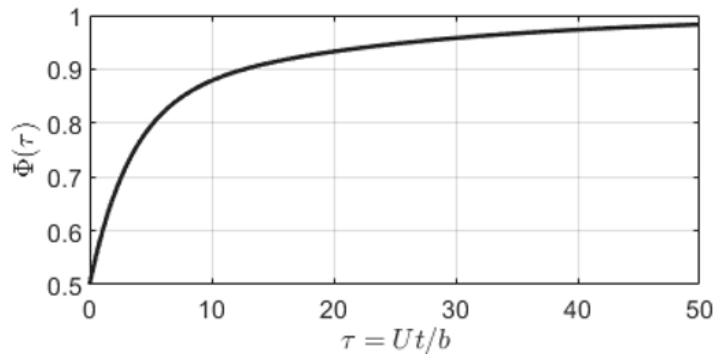


Figure 7 – Wagner's function representation. Source: Author.

Dimitriadis (2017) demonstrates that the term $C(k)W_0$ appearing in Eqs. (2.11) and (2.12) can be equivalently expressed in the time domain through a convolution integral involving $\Phi(t)$, establishing a direct relationship between Theodorsen's and Wagner's formulations. To enable

a state-space representation suitable for time-domain simulations, the exponential terms in Wagner's function are rewritten as additional aerodynamic state variables.

In this approach, two aerodynamic states are introduced per structural degree of freedom. The states w_1 and w_2 are associated with the plunge motion, while w_3 and w_4 are associated with the pitch motion. Their governing equations are given by:

$$\dot{w}_1 = h(t) - \frac{\varepsilon_1 U}{b} w_1(t) \quad (2.16)$$

$$\dot{w}_2 = h(t) - \frac{\varepsilon_2 U}{b} w_2(t) \quad (2.17)$$

$$\dot{w}_3 = \alpha(t) - \frac{\varepsilon_1 U}{b} w_3(t) \quad (2.18)$$

$$\dot{w}_4 = \alpha(t) - \frac{\varepsilon_2 U}{b} w_4(t) \quad (2.19)$$

The first term in each equation corresponds to the associated structural variable for plunge or pitch angle, forming later in Eq. (2.21) the coupling matrix $\mathbf{K}_1$, while the second term represents the decay of the aerodynamic state governed by Wagner's exponential constants, giving rise to the matrix $\mathbf{K}_2$.

Consequently, the aeroelastic equations of motion can be written in the augmented form shown in Eq. (2.20), where $\mathbf{w} = [w_1 \ w_2 \ w_3 \ w_4]^T$ and $\mathbf{y} = [h \ \alpha]^T$. The matrix $\mathbf{K}_0$ defines the influence of the aerodynamic states on the structural equations, while the vector $\mathbf{g}$ accounts for the initial conditions. The remaining terms are identical to those of the quasi-steady formulation, and the system is solved using a time-domain integration with time-varying linearization.

$$(\mathbf{A} + \rho \mathbf{B})\ddot{\mathbf{y}} + (\mathbf{C} + \rho U \mathbf{D})\dot{\mathbf{y}} + [\mathbf{E} + \rho U^2(\Phi(0)\mathbf{F} + \theta \mathbf{D}_2)]\mathbf{y} + \rho U^3 \mathbf{K}_0 \mathbf{w} = \rho U \mathbf{g} \dot{\Phi}(t) \quad (2.20)$$

$$\dot{\mathbf{w}} - \mathbf{K}_1 \mathbf{y} - U \mathbf{K}_2 \mathbf{w} = \mathbf{0} \quad (2.21)$$

$$\mathbf{g} = b \left(h(0) + b \left(\frac{1}{2} - a \right) \alpha(0) \right) \begin{pmatrix} 2\pi \\ -2\pi b(a + 1/2) \end{pmatrix} \quad (2.22)$$

$$\mathbf{F} = \Phi(0) \begin{bmatrix} 0 & 2\pi b \\ 0 & -2\pi b^2(a + \frac{1}{2}) \end{bmatrix} + \theta \begin{bmatrix} 2\pi b & 2\pi b^2(\frac{1}{2} - a) \\ -2\pi b^2(a + \frac{1}{2}) & -2\pi b^3(a + \frac{1}{2})(\frac{1}{2} - a) \end{bmatrix} \quad (2.23)$$

$$\mathbf{K}_0 = \begin{pmatrix} -\Psi_1(\varepsilon_1/b)^2 \\ -\Psi_2(\varepsilon_2/b)^2 \\ \Psi_1 \varepsilon_1 (1 - \varepsilon_1(1/2 - a))/b \\ \Psi_2 \varepsilon_2 (1 - \varepsilon_2(1/2 - a))/b \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} -\varepsilon_1/b & 0 & 0 & 0 \\ 0 & -\varepsilon_2/b & 0 & 0 \\ 0 & 0 & -\varepsilon_1/b & 0 \\ 0 & 0 & 0 & -\varepsilon_2/b \end{pmatrix}$$

$$\theta = \frac{\Psi_1 \varepsilon_1}{b} + \frac{\Psi_2 \varepsilon_2}{b} \quad (2.24)$$

Although Wagner's formulation captures unsteady attached-flow effects and enables time-domain simulations, it remains unable to represent flow separation and dynamic stall. Consequently, the prediction of stall flutter and limit cycle oscillations requires semi-empirical aerodynamic models, which are introduced in the following models.

2.3.3 ONERA Model

The ONERA dynamic stall model, also known as the Petot model, is a semi-empirical aerodynamic formulation designed to reproduce the experimentally observed behavior of aerodynamic coefficients under dynamic stall conditions. Compared to the Leishman–Beddoes model, the ONERA approach is generally simpler and focuses on reconstructing the temporal evolution of lift and pitching moment through a reduced set of state equations.

Two main variants of the ONERA model exist: the original formulation, which considers only pitching motion, and an extended version that accounts for both pitch and plunge. In this work, the extended formulation is adopted, based on the implementations proposed by Malher (2016) and Dimitriadis (2017). Although both approaches follow the same theoretical framework, some differences exist between them. The implementation by Malher (2016) is primarily focused on low-speed flows and large-amplitude motions, which are closely aligned with the objectives of the present research. In contrast, the formulation presented by Dimitriadis (2017) is more general and applicable to a wider range of airspeeds.

Additionally, Dimitriadis (2017) provides a more detailed and systematic description of the model formulation, whereas Malher (2016) does not explicitly describe all implementation parameters. Therefore, the formulation proposed by Dimitriadis (2017) is primarily adopted and described in this work, while specific parameter modifications are introduced based on the approach suggested by Malher (2016).

The general ONERA approach introduces two downwash velocities, defined in Eqs. (2.25) and (2.26). The downwash velocity W_0 corresponds to the effective normal velocity at the aerodynamic center and is consistent with the expression previously introduced in Eq. (2.14), here rewritten using a trigonometric decomposition to accommodate large angles of attack.

$$W_0(t) = \dot{h}(t) \cos \alpha(t) + U \sin \alpha(t) - (x_f - 1/4) \dot{\alpha}(t) \quad (2.25)$$

$$W_1(t) = b \dot{\alpha}(t) \quad (2.26)$$

These downwash velocities are then used to compute the lift force as:

$$l = \rho b \left[(s_l \dot{W}_0 + k_l \dot{W}_1) b + U (\Gamma_1 + \Gamma_2) \right], \quad (2.27)$$

where s_l and k_l are empirical constants. The circulation term Γ_1 represents the contribution of the attached flow and is governed by the first-order differential equation given in Eq. (2.28), while Γ_2 accounts for the contribution of the separated flow and is described by the second-order differential equation in Eq. (2.29).

$$\dot{\Gamma}_1 + \lambda_l \tau \Gamma_1 = \lambda_l \tau c_{l,\alpha} W_0 + \lambda_l \tau W_1 + (\kappa_l c_{l,\alpha} + d_l) \dot{W}_0 + \kappa_l \sigma_l \dot{W}_1 \quad (2.28)$$

$$\ddot{\Gamma}_2 + a_l \tau \dot{\Gamma}_2 + r_l \tau^2 \Gamma_2 = -r_l \tau^2 U \Delta c_l - E_l \tau \dot{W}_0 \quad (2.29)$$

The formulation is expressed in real time t , and therefore the characteristic time constant $\tau = U/b$ is introduced through a variable transformation. The coefficients with subscript l , associated with lift, are obtained empirically and depend on the flow conditions. One practical approach for determining these coefficients is the optimization-based curve-fitting procedure proposed by Malher (2016), although some base values are already proposed in the literature.

Additionally, the term Δc_l represents the difference between the linear lift coefficient c_{l_L} and the true static lift coefficient c_{l_s} as a function of the effective incidence W_0/U , as represented in Figure 8.

$$\Delta c_l = c_{l_L}(W_0/U) - c_{l_s}(W_0/U). \quad (2.30)$$

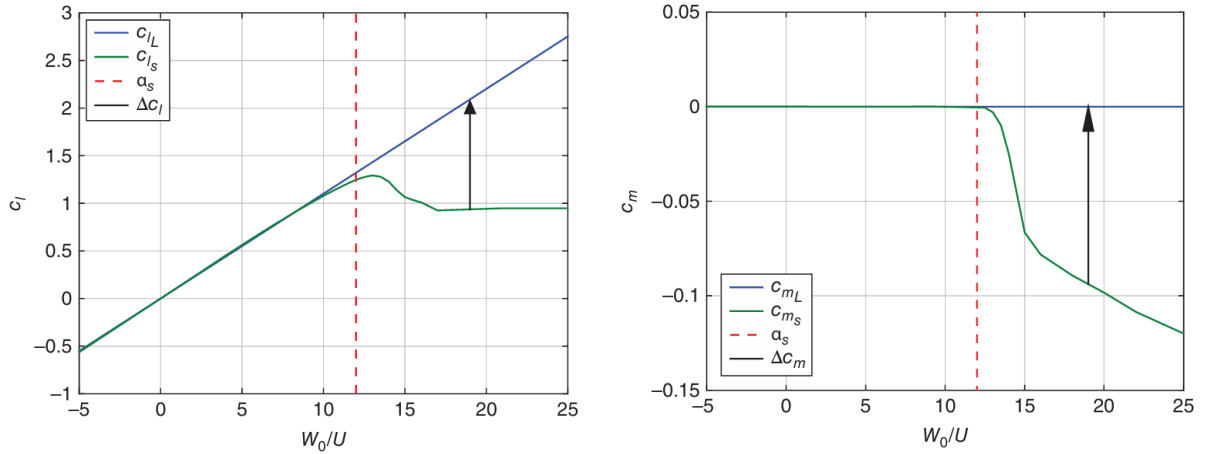
This variation Δc_l is only considered after a time delay $\Delta \tau_{d_l}$ starting from the time the airfoil reaches the static stall angle, otherwise $\Delta c_l = 0$.

Furthermore, the aerodynamic pitching moment is computed in an analogous manner, as:

$$m_{xf} = 2\rho b^2 (U^2 c_{m_L}(W_0/U) + ((\bar{\sigma}_m + d_m) b \dot{W}_0 + \sigma_m U W_1 + s_m b \dot{W}_1 + U \Gamma_3)) \quad (2.31)$$

where the coefficients with subscript m are empirical parameters related to the pitching moment. The circulation term Γ_3 represents the contribution of the stalled flow and is governed by the second-order differential equation given in Eq. (2.32). The attached-flow contribution is directly estimated using the linear moment coefficient $c_{m_L}(W_0/U)$.

$$\ddot{\Gamma}_3 + a_m \tau \dot{\Gamma}_3 + r_m \tau^2 \Gamma_3 = -r_m \tau^2 U \Delta c_m - E_m \tau \dot{W}_0 \quad (2.32)$$

Figure 8 – Representation of Δc_l and Δc_m . Source: Dimitriadis (2017).

As for the lift, the difference between the linear and static pitching moment coefficients is introduced through the Eq. (2.33) after a time delay $\Delta\tau_{d_m}$.

$$\Delta c_m = c_{mL}(W_0/U) - c_{mS}(W_0/U), \quad (2.33)$$

For the flat-plate configuration considered, the static aerodynamic curves c_{l_s} and c_{m_s} were obtained experimentally by Malher (2016) and implemented in the present work using linear interpolation. The resulting static lift and moment characteristics are shown in Fig. 9.

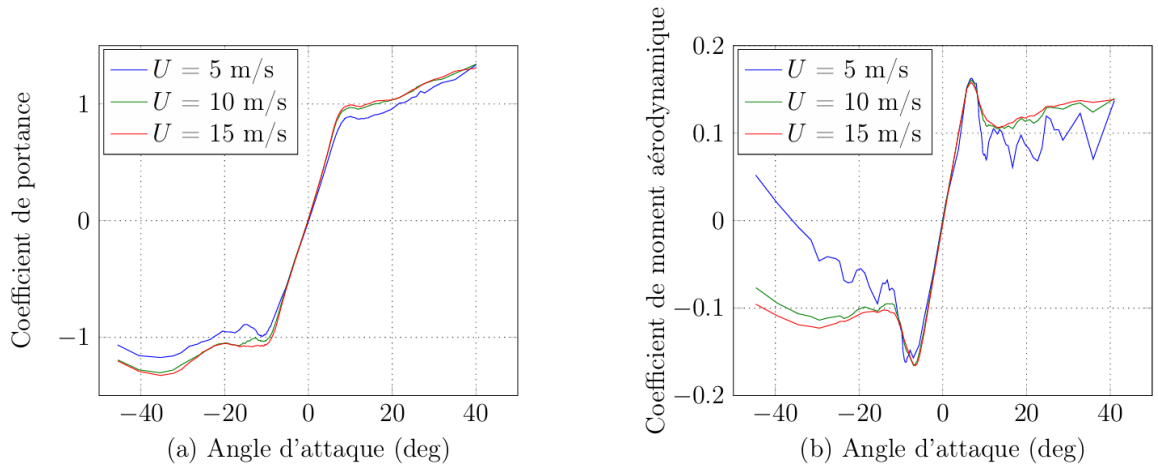


Figure 9 – Experimental lift and moment coefficients for a flat plate. Source: Malher (2016).

For literature comparison, the formulation proposed by Malher (2016) employs differential equations similar to those presented by Dimitriadis (2017). However, these equations are not expressed in terms of the circulations Γ , but rather directly in terms of the aerodynamic coefficients $C_{L,1}$, $C_{L,2}$, $C_{M,1}$, and $C_{M,2}$, as well as their corresponding derivatives. A further fundamental difference is that the pitching moment is modeled using two differential equations, whereas Dimitriadis (2017) adopts a single equation derived from Γ_3 . This separation in the

formulation of Malher (2016) aims to provide a more accurate representation of the linear and nonlinear aerodynamic contributions to the pitching moment.

In addition to the model inputs, several ONERA model coefficients are assumed to vary as functions of the lift deviation Δc_l and moment deviation Δc_m in order to account for nonlinear and separated-flow effects. These dependencies are defined by the relations given in Eq. (2.34), while the corresponding reference coefficient values are listed in Table 1. Malher (2016) provides similar expressions for the lift-related coefficients, whereas the moment-related coefficients are not directly formulated.

$$\begin{aligned}
 a_l &= a_{0_l} + a_{2_l} \Delta c_l^2, & a_m &= a_{0_m} + a_{2_m} \Delta c_l^2, \\
 r_l &= (r_{0_l} + r_{2_l} \Delta c_l^2)^2, & r_m &= (r_{0_m} + r_{2_m} \Delta c_l^2)^2, \\
 \sigma_l &= \sigma_{0_l} + \sigma_{2_l} \Delta c_l^2, & \sigma_m &= \sigma_{0_m} + \sigma_{2_m} \Delta c_l^2, \\
 E_l &= E_{2_l} \Delta c_l^2, & E_m &= E_{2_m} \Delta c_m^2, \\
 d_l &= \sigma_{2_l} |\Delta c_l|, & d_m &= \sigma_{2_m} |\Delta c_l|
 \end{aligned} \tag{2.34}$$

Table 1 – Proposed ONERA model parameters for lift and pitch moment. Source: Malher (2016).

Lift-related parameters									
	λ^L	κ^L	σ_0^L	r_0^L	a_0^L	σ_2^L	r_2^L	a_2^L	E_2^L
Value	0.119	0.8	0.1	0.15	0.14	−0.005	0.09	0.26	−0.004

Moment-related parameters									
	λ^M	κ^M	σ_0^M	r_0^M	a_0^M	σ_2^M	r_2^M	a_2^M	E_2^M
Value	0.1	0.43	0.15	0.19	0.4	−0.1	0	0.08	0

Finally, the ONERA model implementation by Malher (2016) is calibrated for a flat plate using experimental data. Figure 10 presents a comparison between the measured lift and moment coefficients under dynamic stall conditions for different reduced frequencies k and the corresponding results obtained from the model implementation for an imposed sinusoidal pitch motion. Overall, the model shows good agreement with the experimental data in terms of lift and moment prediction, whereas some local discrepancies are observed in the moment case. Additionally, this example illustrates the relationship between the reduced frequency and the linear behavior, which becomes predominantly nonlinear at higher frequencies.

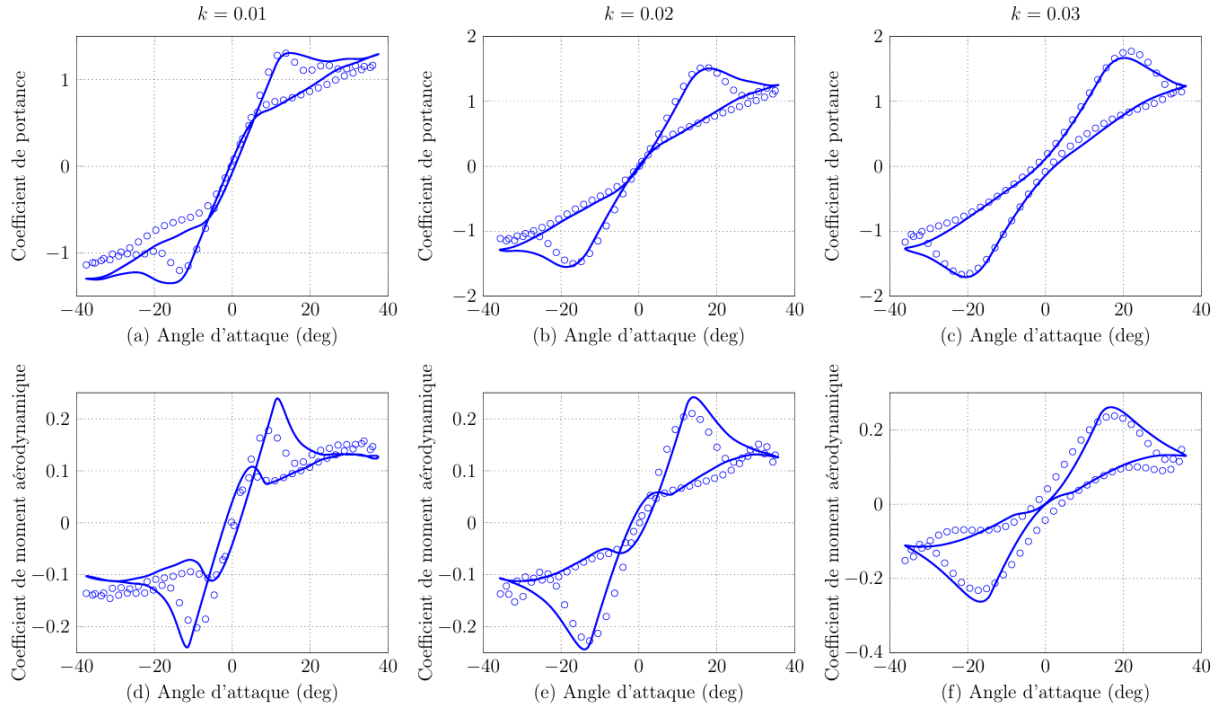


Figure 10 – Lift coefficient (a, b, c) and aerodynamic moment coefficient (d, e, f) of the flat plate as functions of the angle of attack for an amplitude of 36 degrees and three reduced frequencies: 0.01 (a, d), 0.02 (b, e), and 0.03 (c, f). Solid line: represents ONERA model with the parameters from Table 8. Circles: represents experimental measurements. Source: Malher (2016).

2.3.4 Leishman–Beddoes model

The Leishman–Beddoes (LB) model establish one approach for representing dynamic stall by introducing states for multiple physical mechanisms associated with unsteady separated flow. Several variants of this model exist in the literature, reflecting different assumptions and formulations of the phenomena. The implementation adopted in this work follows the formulation proposed by Santos and Marques (2021b), which introduces improvements aimed at better reproducing low-speed experimental results and provides an open-source implementation available on GitHub.

The model has been previously validated for the symmetric NACA 0012 airfoil under forced oscillations, covering reduced frequencies in the range $0.01 < k < 0.3$, Mach numbers $0.035 < M < 0.3$, and pitch amplitudes up to $\pm 30^\circ$. This wide applicability is enabled by the adoption of an interpolation method for the model parameters according to the flow conditions. In addition, Santos and Marques (2021b) compared the proposed formulation with the model introduced by Sheng, Galbraith and Coton (2008), demonstrating improved agreement with experimental dynamic stall data. The main differences between these formulations lie in the number of aerodynamic states, the treatment of time delays, and the stall onset and suppression criteria. In the present implementation, a total of 14 aerodynamic states are employed, whereas Sheng, Galbraith and Coton (2008) adopts 9 aerodynamic states.

The aerodynamic formulation is presented here as an overview of the main modeling steps, while the complete set of governing equations can be found in Santos and Marques (2021b). A summary of the adopted aerodynamic states is given in Table 2. The first eight states represent the unsteady attached flow and are directly related to the effective angle of attack $\bar{\alpha}$ and the nondimensional unsteady pitch rate q . These quantities are defined following Eqs. (2.35) to (2.36) and can be interpreted as alternative representations of the kinematic quantities previously introduced in Eqs. (2.25) and (2.26).

$$\bar{\alpha} = \tan^{-1} \left(\frac{W_0(t)}{U \cos \alpha - \dot{h} \sin \alpha} \right), \quad (2.35)$$

$$q = \frac{\dot{\bar{\alpha}} c}{U}, \quad (2.36)$$

Table 2 – Description of Leishman–Beddoes aerodynamic states.

States	Physical interpretation
x_1 to x_8	Unsteady attached-flow aerodynamic states (linear)
x_9	$\bar{\alpha}$ delayed by a time constant T_α
x_{10}	Delayed reduced pitch-rate parameter R (stall onset)
x_{11}	Delayed reduced pitch-rate parameter R (stall suppression)
x_{12} to x_{14}	Lagged separation-point dynamics for c_n , c_m , and c_c

Unlike the previously discussed aerodynamic models, which directly compute lift and pitching moment, the Leishman–Beddoes model is formulated in the airfoil reference coordinates. The aerodynamic loads are decomposed into normal force (c_n), chordwise force (c_c), and quarter-chord pitching moment (c_m) coefficients. The lift coefficient c_l is subsequently obtained by trigonometric projection of c_n and c_c .

The attached-flow states allow the computation of the circulatory coefficients c_n^C and c_m^C , which are related to Wagner’s unsteady aerodynamic formulation with compressibility corrections. In addition, the impulsive components c_n^I and c_m^I are included to account for non-circulatory effects, typically associated with piston-theory concepts.

The ninth state represents the effective angle of attack delayed in time, while the tenth and eleventh states introduce delays in the reduced pitch-rate parameter R . The parameter R is defined according Eq. (2.37), while state x_{10} is associated with stall onset, whereas x_{11} employs a shorter time constant and is used to model stall suppression.

$$R = \min \left(\frac{|r|}{r_0}, 1 \right), \quad r = \frac{q}{2}. \quad (2.37)$$

Nonlinear effects associated with trailing-edge separation are incorporated through the introduction of the separation-point parameter f , which depends on the instantaneous angle

of attack. Experimental observations indicate that the different aerodynamic loads respond to separation with distinct time delays, motivating the introduction of the last three states, x_{12} to x_{14} . These states combine the original LB delay formulation with additional unsteady boundary-layer effects. At this stage, the increased complexity of the Leishman–Beddoes model relative to the ONERA model becomes apparent, both in the number of empirical parameters and in the presence of multiple conditional statements used to represent different flow regimes.

Using the circulatory and impulsive contributions together with the lagged separation states, the nonlinear unsteady aerodynamic coefficients c_n^f , c_m^f , and c_c^f are obtained.

Finally, dynamic stall vortex effects are modeled through the formation and shedding of leading-edge vortices (LEVs). The formulation assumes that the vortex-induced loads are proportional to the difference between the lagged and steady separation points. Since multiple vortices may be shed under certain conditions, the model allows for up to three simultaneous vortices, with the resultant contributions computed according Eq. (2.38).

$$c_n^v = c_{n_1}^v + c_{n_2}^v + c_{n_3}^v, \quad c_m^v = c_{m_1}^v + c_{m_2}^v + c_{m_3}^v \quad (2.38)$$

The total aerodynamic coefficients are finally obtained as the sum of the contributions of the separated condition, the impulsive condition and the vortices condition.

$$c_n = c_n^f + c_n^I + c_n^v, \quad (2.39)$$

$$c_m = c_m^f + c_m^I + c_m^v, \quad (2.40)$$

$$c_c = c_c^f. \quad (2.41)$$

2.4 Numerical Solution

The governing equations presented for modeling the aeroelastic dynamics are expressed in the form of ordinary differential equations. Since the objective of this work is to obtain time-domain responses of the system for given initial conditions, the use of numerical time-integration methods is required.

The numerical solution strategy adopted is based on a state-space formulation of the aeroelastic problem. In this framework, a state vector is defined to represent the instantaneous condition of the aeroelastic system, including structural displacements, velocities, and, when applicable, aerodynamic states. The general representation employed in this work consists of expressing the time derivative of the state vector as a function of its current value, through a linear matrix $\mathbf{Q}$, as presented in Eq. (2.42).

$$\dot{\mathbf{x}} = \mathbf{Q}\mathbf{x} \quad (2.42)$$

Once the system equations are written in state-space form, several numerical integration techniques can be employed to advance the solution in time. In this work, an explicit Runge–Kutta method is adopted due to its robustness, accuracy, and favorable computational cost for problems involving moderately stiff systems.

The time integration is performed using the MATLAB function `ode45`, which implements a variable-step Runge–Kutta scheme of orders 4 and 5. This method automatically adjusts the time step based on local truncation error estimates, providing an efficient compromise between accuracy and computational effort. The use of `ode45` is well suited for the nonlinear aeroelastic models considered in this study (The MathWorks, Inc., 2026).

METHODOLOGY

IN this chapter, the methodology adopted for implementing the numerical models is presented. The general problem formulation is introduced, followed by the definition of the geometric parameters of the aeroelastic system. A validation-oriented geometry is adopted, aiming at future comparisons between numerical results and experimental data available in the literature.

3.1 Study Case

As discussed in the previous chapters, the main objective of this work is to evaluate non-linear aeroelastic effects associated with dynamic stall in a flat-plate configuration. In this context, the classical pitch–plunge aeroelastic model is adopted, as it represents a configuration capable of capturing the essential coupling between structural motion and unsteady aerodynamics.

The problem considered consists of a rigid flat plate with two structural degrees of freedom: plunge and pitch. The plunge displacement is denoted by h and is defined as positive in the downward direction, while the pitch angle is denoted by α and is considered positive when the leading edge rotates upward. Figure 11 illustrates the adopted pitch–plunge configuration and reference frame.

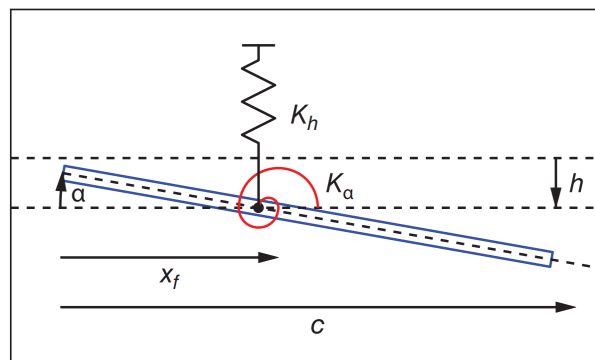


Figure 11 – Pitch–plunge aeroelastic model. Source: Dimitriadis (2017).

Regarding the geometric parameters, the flat plate is characterized by its chord length c and span. The elastic center, defined as the point where both translational and rotational springs are attached, is located at a longitudinal position x_f measured from the leading edge.

The structural model also includes inertial properties, namely the mass m and the mass moment of inertia I_α about the elastic center. The mass distribution of the plate is assumed to be uniform, resulting in the center of gravity being located at the mid-chord position. Consequently, when the elastic center does not coincide with the center of gravity, an inertial coupling is introduced. This offset is represented by the static unbalance term S , as described in Section 2.1, and plays a fundamental role in aeroelastic coupling.

The elastic center is connected to two linear springs, being a translational spring with stiffness K_h , associated with plunge motion, and a rotational spring with stiffness K_α , associated with pitch motion. Structural damping effects are also included through linear damping coefficients C_h and C_α , which are assumed to act at the same location as the elastic springs.

Another important set of parameters is the uncoupled natural frequencies of the structure, namely the plunge natural frequency ω_h and the pitch natural frequency ω_α . Their definitions follow the formulation presented in Section 2.1. These quantities are functions of the structural stiffness and inertial properties and represent the natural frequencies of the system when aerodynamic coupling is neglected.

Depending on the objective of each numerical investigation, the geometric and structural parameters can be adjusted to match specific experimental configurations available in the literature. In this work, two reference sets of parameters are considered. Table 3 presents the structural parameters corresponding to the low-speed wind tunnel experiments reported by Amandolese, Michelin and Choquel (2011), while Table 4 summarizes the parameters associated with the experimental configuration proposed by Malher (2016). These configurations serve as validation cases for the numerical models implemented in this study.

Table 3 – Structural parameters in SI units used by Amandolese, Michelin and Choquel (2011).

Chord		Span	x_f	ω_h	ω_α		
0.035		0.225	0.0147	44.26	56.55		
m	S	I_α	C_h	C_α	K_h	K_α	
0.304	8.48×10^{-4}	4.66×10^{-5}	5.38×10^{-2}	7.91×10^{-5}	595.6	0.149	

Table 4 – Structural parameters in SI units used by Malher (2016).

Chord		Span	x_f	ω_h	ω_α		
0.035		0.225	0.0175	26.94	26.03		
m	S	I_α	C_h	C_α	K_h	K_α	
0.389	1.0×10^{-3}	2.11×10^{-4}	0.126	1.65×10^{-4}	282.3	0.143	

3.2 Models Adaptations

For all aerodynamic models presented in Section 2.3, the corresponding theoretical formulations are extracted from the literature and implemented numerically in MATLAB. Depending on the characteristics of each model, certain adaptations are required to ensure compatibility with the numerical time-integration adopted in this work. The following subsections describe the specific implementation details for each aerodynamic approach.

3.2.1 Quasi-steady Adaptation

The quasi-steady aerodynamic model is implemented according to the original formulation given by Eq. (2.7), as detailed in Section 2.3.1. By assuming attached flow and neglecting unsteady aerodynamic effects, this model results in a linear relationship between the aerodynamic loads and the instantaneous kinematic states.

In order to adapt this formulation to the numerical integration described in Section 2.42, the governing equation is rewritten in state-space form. For this purpose, the second-order system is first rearranged by isolating the acceleration vector $\ddot{\mathbf{y}}$, giving Eq. (3.1).

$$\ddot{\mathbf{y}} = -(\mathbf{A} + \rho\mathbf{B})^{-1}(\mathbf{C} + \rho\mathbf{UD})\dot{\mathbf{y}} - (\mathbf{A} + \rho\mathbf{B})^{-1}(\mathbf{E} + \rho\mathbf{U}^2\mathbf{F})\mathbf{y}, \quad \mathbf{y} = \begin{bmatrix} h \\ \alpha \end{bmatrix} \quad (3.1)$$

Next, the state vector is defined using the structural degrees of freedom and their first time derivatives, namely plunge and pitch displacements and velocities. The resulting state-space representation is given by

$$\frac{d}{dt} \begin{bmatrix} \dot{h} \\ \dot{\alpha} \\ h \\ \alpha \end{bmatrix} = \mathbf{Q} \begin{bmatrix} \dot{h} \\ \dot{\alpha} \\ h \\ \alpha \end{bmatrix}. \quad (3.2)$$

From this formulation, the associated system matrix $\mathbf{Q}$ can be directly obtained and is expressed as

$$\mathbf{Q} = \begin{bmatrix} -(\mathbf{A} + \rho\mathbf{B})^{-1}(\mathbf{C} + \rho\mathbf{UD}) & -(\mathbf{A} + \rho\mathbf{B})^{-1}(\mathbf{E} + \rho\mathbf{U}^2\mathbf{F}) \\ \mathbf{I}_{2 \times 2} & \mathbf{0}_{2 \times 2} \end{bmatrix}, \quad (3.3)$$

where $\mathbf{I}_{2 \times 2}$ is the identity matrix and $\mathbf{0}_{2 \times 2}$ is a null matrix. This linear state-space system can then be directly integrated in time using ode45 and assuming known initial conditions.

In addition to time integration, the state matrix $\mathbf{Q}$ provides valuable information regarding the dynamic characteristics of the system. The eigenvalues of $\mathbf{Q}$, obtained using the MATLAB

function $\mathbf{eig}$, represent the system poles and are directly related to the aeroelastic modes of the structure. From a control theory perspective, the stability of the system is determined by the real part of the eigenvalues. A mode is stable when the real part of its eigenvalue is negative, while a positive real part indicates an unstable mode. In the context of this work, flutter onset is identified when a pair of complex-conjugate eigenvalues crosses the imaginary axis, that is, when the real part becomes zero and then positive as the airspeed increases.

The complex eigenvalues are associated with the oscillation frequency of each mode, such that the modal frequency is given by $\omega = \text{abs}(\lambda)$. The modal damping ratio ζ can be computed from the eigenvalues according to $\zeta = -\text{Re}(\lambda)/\omega$. This methodology of identification is used for extracting the flutter specifications and to build the *root locus* for this and the following models.

3.2.2 Unsteady Adaptation

For the unsteady aerodynamic solution, Wagner's model is adopted, as it allows a time-domain representation of the transient aerodynamic load response. The formulation implemented in this work is based on Eqs. (2.20) and (2.21). Similarly to the quasi-steady case, the second-order derivatives are isolated, leading to the formulation presented in Eq. (3.4).

$$\ddot{\mathbf{y}} = -(\mathbf{A} + \rho\mathbf{B})^{-1} \left[(\mathbf{C} + \rho U\mathbf{D})\dot{\mathbf{y}} + (\mathbf{E} + \rho U^2\mathbf{F})\mathbf{y} + \rho U^3\mathbf{K}_0\mathbf{w} - \rho U\mathbf{g}\dot{\Phi}(t) \right] \quad (3.4)$$

It is important to note that the initial conditions are explicitly considered in the formulation through the vector $\mathbf{g}$, which contains information related to the initial pitch and plunge displacements. This term accounts for the effect of the Wagner function derivative $\dot{\Phi}(t)$ on the initial transient aerodynamic response.

Also, the unsteady aerodynamic contribution is represented through a set of additional aerodynamic states, whose temporal evolution is governed by the following first-order differential equation:

$$\dot{\mathbf{w}} = \mathbf{K}_1\mathbf{y} + U\mathbf{K}_2\mathbf{w}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \quad (3.5)$$

Since aerodynamic states are introduced in addition to the structural kinematic variables, the complete state vector used for time integration is formed by the combination of pitch and plunge variables and the Wagner aerodynamic states. As a result, the system is described by a total of eight states, which is twice the number of states employed in the quasi-steady formulation.

The resulting state-space representation consists of a linear term defined by the matrix $\mathbf{Q}$ and an additional time-dependent input term associated with the initial conditions and the Wagner function derivative $\dot{\Phi}(t)$.

$$\frac{d}{dt} \begin{bmatrix} \dot{h} \\ \dot{\alpha} \\ h \\ \alpha \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \mathbf{Q} \begin{bmatrix} \dot{h} \\ \dot{\alpha} \\ h \\ \alpha \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} + \begin{bmatrix} (\mathbf{A} + \rho\mathbf{B})^{-1}\mathbf{g} \\ \mathbf{0}_{6 \times 1} \end{bmatrix} \rho U \dot{\Phi}(t) \quad (3.6)$$

Following this formulation, the global linear state matrix $\mathbf{Q}$ can be assembled as shown in Eq. (3.7). Although the presence of the additional input term related to the initial conditions, the matrix $\mathbf{Q}$ remains the most relevant element for characterizing the dynamic behavior of the system, as it governs the modal response. In particular, the flutter frequency and modal damping can be obtained through the analysis of the eigenvalues of $\mathbf{Q}$, following the same procedure adopted for the quasi-steady formulation.

$$\mathbf{Q} = \begin{bmatrix} -(\mathbf{A} + \rho\mathbf{B})^{-1}(\mathbf{C} + \rho\mathbf{U}\mathbf{D}) & -(\mathbf{A} + \rho\mathbf{B})^{-1}(\mathbf{E} + \rho\mathbf{U}^2\mathbf{F}) & -(\mathbf{A} + \rho\mathbf{B})^{-1}\rho\mathbf{U}^3\mathbf{K}_0 \\ \mathbf{I}_{2 \times 2} & \mathbf{0}_{2 \times 2} & \mathbf{0}_{2 \times 4} \\ \mathbf{0}_{4 \times 2} & \mathbf{K}_1 & \mathbf{U}\mathbf{K}_2 \end{bmatrix} \quad (3.7)$$

3.2.3 ONERA Model Adaptation

Extending the analysis to aerodynamic models capable of representing dynamic stall effects, the ONERA model is implemented in a progressive manner. Initially, the aerodynamic formulation is considered independently, without coupling it to the structural degrees of freedom h and α . This preliminary stage enables the verification and validation of the aerodynamic model before its integration into the complete aeroelastic problem.

Similarly to the original formulation, the initial validation of the aerodynamic model is performed assuming an imposed pitch motion. The pitch angle is prescribed as a sinusoidal function of time, centered at a mean angle α_0 , with amplitude $\Delta\alpha$ and angular frequency ω , such as Eq. (3.8).

$$\alpha(t) = \alpha_0 + \Delta\alpha \sin(\omega t) \quad (3.8)$$

This imposed motion has the objective to verify the correct implementation of the ONERA model and is later removed when the pitch and plunge degrees of freedom are introduced in the fully coupled aeroelastic formulation.

Following the general solution procedure, the kinematic information associated with pitch and plunge is transformed into equivalent downwash components and their respective time derivatives, which act as inputs to the model. In particular, the normalized downwash component W_0/U plays the role of an effective angle of attack, as it represents the ratio between the vertical component of the relative flow velocity and the freestream velocity. This interpretation justifies its subsequent use in the evaluation of aerodynamic coefficients c_l and c_m .

It is also noted that the expressions for the downwash include second-order time derivatives of the structural variables, which would require a decision of implicit or explicit formulation when considering the free-motion aeroelastic case. The downwash components W_0 and W_1 , together with their time derivatives, are defined as follows:

$$\begin{aligned} W_0 &= U \left(\frac{\dot{h}}{U} + \alpha - \left(x_f - \frac{b}{2} \right) \frac{\dot{\alpha}}{U} \right), \\ \dot{W}_0 &= U \left(\frac{\ddot{h}}{U} \cos \alpha + \dot{\alpha} \cos \alpha - \frac{\dot{h}\dot{\alpha}}{U} \sin \alpha - \left(x_f - \frac{b}{2} \right) \frac{\ddot{\alpha}}{U} \right), \\ W_1 &= b \dot{\alpha}, \\ \dot{W}_1 &= b \ddot{\alpha}. \end{aligned} \quad (3.9)$$

Once these quantities are defined, the ONERA circulation equations, given in Eqs. (2.28), (2.29), and (2.32), can be rearranged by isolating the highest-order time derivatives. This manipulation allows the aerodynamic model to be recast into a state-space formulation. At this stage, the input vector $\mathbf{u}$ is composed of the downwash components and the aerodynamic coefficients variations of lift and moment. The resulting state-space representation of the ONERA aerodynamic model is in expression (3.10).

$$\dot{\mathbf{x}} = \mathbf{A}_{\text{onera}} \mathbf{x} + \mathbf{B}_{\text{onera}} \mathbf{u} \quad \rightarrow \quad \frac{d}{dt} \begin{bmatrix} \Gamma_1 \\ \dot{\Gamma}_2 \\ \Gamma_2 \\ \dot{\Gamma}_3 \\ \Gamma_3 \end{bmatrix} = \mathbf{A}_{\text{onera}} \begin{bmatrix} \Gamma_1 \\ \dot{\Gamma}_2 \\ \Gamma_2 \\ \dot{\Gamma}_3 \\ \Gamma_3 \end{bmatrix} + \mathbf{B}_{\text{onera}} \begin{bmatrix} W_0 \\ W_1 \\ \dot{W}_0 \\ \dot{W}_1 \\ \Delta c_l \\ \Delta c_m \end{bmatrix} \quad (3.10)$$

The state matrix $\mathbf{A}_{\text{onera}}$ and the corresponding input matrix $\mathbf{B}_{\text{onera}}$ are defined according to the representative model parameters. To better match the numerical results with the expected physical behavior, a sensitivity analysis is included as part of the methodology for model calibration and validation. This step is necessary, since the adopted parameter values have a

significant influence on the final results.

$$\mathbf{A}_{\text{onera}} = \begin{bmatrix} -\lambda_l \frac{U}{b} & 0 & 0 & 0 & 0 \\ 0 & -a_l \frac{U}{b} & -r_l \left(\frac{U}{b} \right)^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_m \frac{U}{b} & -r_m \left(\frac{U}{b} \right)^2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (3.11)$$

$$\mathbf{B}_{\text{onera}} = \begin{bmatrix} \lambda_l \frac{U}{b} c_{l,\alpha} & \lambda_l \frac{U}{b} & (\kappa_l c_{l,\alpha} + d_l) & \kappa_l \sigma_l & 0 & 0 \\ 0 & 0 & -E_l \frac{U}{b} & 0 & -r_l \left(\frac{U}{b} \right)^2 U & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -E_m \frac{U}{b} & 0 & 0 & -r_m \left(\frac{U}{b} \right)^2 U \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.12)$$

Next, focusing on the aerodynamic coefficient terms in the input vector, it is important to note that the difference between the linear and static aerodynamic coefficients depends on the adopted stall delay τ_d . This variation is only taken into account after the effective incidence exceeds the stall angle α_d . Once this condition is met, the contribution is introduced after a prescribed delay, as defined by the stall delay time. The implemented logic for the lift coefficient variation Δc_l is presented in Eq. (3.13), and the same formulation is applied analogously to the pitching moment coefficient variation Δc_m .

$$\Delta c_l(W_0/U) = \begin{cases} c_{l_L}(W_0/U) - c_{l_s}(W_0/U), & |W_0/U| \geq \alpha_d \text{ and } \tau - \tau_{\text{stall}} > \Delta \tau_d, \\ 0, & \text{otherwise.} \end{cases} \quad (3.13)$$

The solution of the aerodynamic state–space system is obtained numerically using a Runge–Kutta time integration scheme. Once the circulation states are computed, the aerodynamic loads l and m are evaluated as functions of the previously defined variables and model parameters.

$$l = \rho b \left[(s_l \dot{W}_0 + k_l \dot{W}_1) b + U (\Gamma_1 + \Gamma_2) \right], \quad (3.14)$$

$$m = 2\rho b^2 (U^2 c_{m_L}(W_0/U) + ((\bar{\sigma}_m + d_m) b \dot{W}_0 + \sigma_m U W_1 + s_m b \dot{W}_1 + U \Gamma_3)) \quad (3.15)$$

Following this procedure, the transient aerodynamic lift and pitching moment associated with an imposed motion can be obtained, allowing the dynamic response of the ONERA aerodynamic model.

In the second step, the implementation is extended to the aeroelastic problem by introducing the structural degrees of freedom associated with pitch and plunge. At this stage, the aerodynamic model is fully coupled with the structural dynamics. The motion response is obtained by isolating the acceleration terms in Eq. (2.1), resulting in expressions for the plunge and pitch accelerations as functions of the aerodynamic lift and pitching moment.

$$\begin{pmatrix} \ddot{h} \\ \ddot{\alpha} \end{pmatrix} = -\mathbf{A}^{-1}\mathbf{C} \begin{pmatrix} \dot{h} \\ \dot{\alpha} \end{pmatrix} - \mathbf{A}^{-1}\mathbf{E} \begin{pmatrix} h \\ \alpha \end{pmatrix} + \mathbf{A}^{-1} \begin{pmatrix} -l \\ m_{xf} \end{pmatrix} \quad (3.16)$$

An important aspect of this formulation is that the downwash components are now directly defined as functions of the pitch and plunge states. In addition, the presence of the second-order time derivatives $\ddot{h}$ and $\ddot{\alpha}$ requires special attention in the numerical implementation. The time integration scheme adopted at this stage is explicit, as the accelerations at each time step are evaluated using state information from the previous step. This approach avoids the need for implicit algebraic solutions, which would otherwise increase the complexity of the formulation.

The final coupled aeroelastic system is represented by a total of nine state variables, defined as

$$\mathbf{x} = \left[\Gamma_1 \quad \dot{\Gamma}_2 \quad \Gamma_2 \quad \dot{\Gamma}_3 \quad \Gamma_3 \quad \dot{h} \quad h \quad \dot{\alpha} \quad \alpha \right]^T. \quad (3.17)$$

3.2.4 Leishman–Beddoes Model Adaptation

The Leishman–Beddoes dynamic stall model is the semi-empirical approach that introduces a set of dedicated states to represent the main physical mechanisms associated with dynamic stall. Among the various formulations available in the literature, the model adopted in this work is based on the formulation proposed by Santos and Marques (2021b), which includes specific improvements for low-speed applications, the same operational regime considered in the present study.

In addition to the published theoretical formulation, the authors provide a validated MATLAB reference implementation through a public GitHub repository (SANTOS; MARQUES, 2021a). It is important to remark that this code computes aerodynamic coefficients with dynamic stall effects for an imposed sinusoidal pitch motion, and is not originally designed to perform aeroelastic simulations with free pitch and plunge degrees of freedom. The contribution of the present work is therefore to complement the original aerodynamic model by extending its application to a fully coupled aeroelastic framework, using the validated aerodynamic coefficients as a starting point.

The LB model parameters are extensively calibrated using publicly available experimental data from NASA for a NACA 0012 airfoil, being an interpolation table adopted. In this work, the same airfoil geometry is adopted as an approximate representation of a flat plate, based on its null camber symmetry. Nevertheless, differences related to stall angle prediction and LEV development may arise, and these aspects are taken into account in the critical analysis of the results.

In its original formulation, the model is composed of fourteen aerodynamic states. In the present implementation, the structural degrees of freedom in pitch and plunge, along with their first derivatives, are incorporated into the system, resulting in a total of eighteen states. As in the previous aeroelastic formulations, the primary aerodynamic outputs of the model are the lift l and pitching moment m , which are subsequently used to compute the structural accelerations through the dynamic equilibrium equation given in Eq. (3.16).

The resulting state vector of the coupled system is defined as

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_{13} & x_{14} & \dot{h} & \dot{\alpha} & h & \alpha \end{bmatrix}^T. \quad (3.18)$$

The numerical solution of the coupled system is performed using a Runge–Kutta–Fehlberg integration scheme, with event detection to properly capture discontinuities inherent to the dynamic stall model. Since several aerodynamic parameters depend strongly on the instantaneous flow state, the simulations are initialized by imposing one complete oscillation cycle of the airfoil, after which the fully coupled free-motion aeroelastic response is computed.

Finally, it is worth noting that the LB model primarily depends on the equivalent angle of attack $\bar{\alpha}$ and the pitch rate q as input variables. These quantities are alternative representations of the downwash components W_0 and W_1 previously employed in the ONERA model and are defined in Eqs. (2.35) and (2.36). Their time derivatives, $\dot{\bar{\alpha}}$ and $\dot{q}$, depend explicitly on the pitch and plunge accelerations, $\ddot{\alpha}$ and $\ddot{h}$. Consequently, an explicit numerical strategy is again adopted, in which the accelerations from the previous time step are used to evaluate these quantities at the current iteration.

3.3 Validation Cases

After the implementation of the aerodynamic and aeroelastic models, it is important to define appropriate validation criteria and comparison bases for the next obtained results. The validation cases are therefore selected according to the main objectives of the developed tools and the expected experimental behavior. For each modeling approach, its applicability, limitations, and underlying assumptions are evaluated. The primary reference used for validation is the experimental and numerical study presented in Amandolese, Michelin and Choquel (2011).

Figure 12 presents the experimental data points together with the evolution of the pitch and plunge modal frequencies as a function of airspeed, up to the onset of the LCO, where a

single dominant frequency is observed. The figure also shows, in its lower part, the evolution of the real part of the system poles, highlighting the airspeed at which one of the poles becomes positive. This transition characterizes the loss of stability and can be directly identified through the theoretical models employed in this work.

The comparison between the implemented numerical methods and the reference data is performed by evaluating the system poles in cases where a direct state matrix $\mathbf{Q}$ is available. For the more complex aerodynamic models, in which an explicit global state matrix cannot be readily constructed, the modal frequencies are obtained through a spectral analysis of the time-domain response using the Fourier transform.

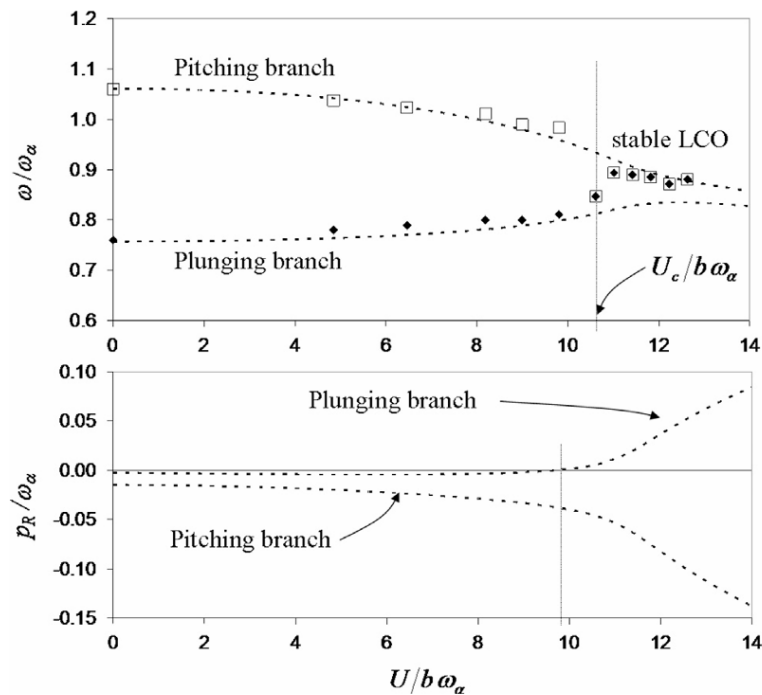


Figure 12 – Frequency and real part of the poles for the reference pitch–plunge problem. Source: Aman-dolese, Michelin and Choquel (2011).

Figure 13 illustrates the hysteretic behavior associated with the LCO regime. Below the critical airspeed, no sustained oscillations are observed, as the damping exceeds the excitation. Once the critical airspeed is exceeded and LCO is observed, further increases in velocity lead to a nearly linear growth of the oscillation amplitudes in both pitch and plunge. After reaching the maximum test airspeed, the experiment proceeds by decreasing the airspeed. In this descending branch, the system initially follows the same amplitude levels but maintains the LCO at airspeeds lower than the critical value. This behavior indicates that, once established, the LCO provides sufficient excitation to sustain oscillations even at conditions where they would not develop from a static initial state.

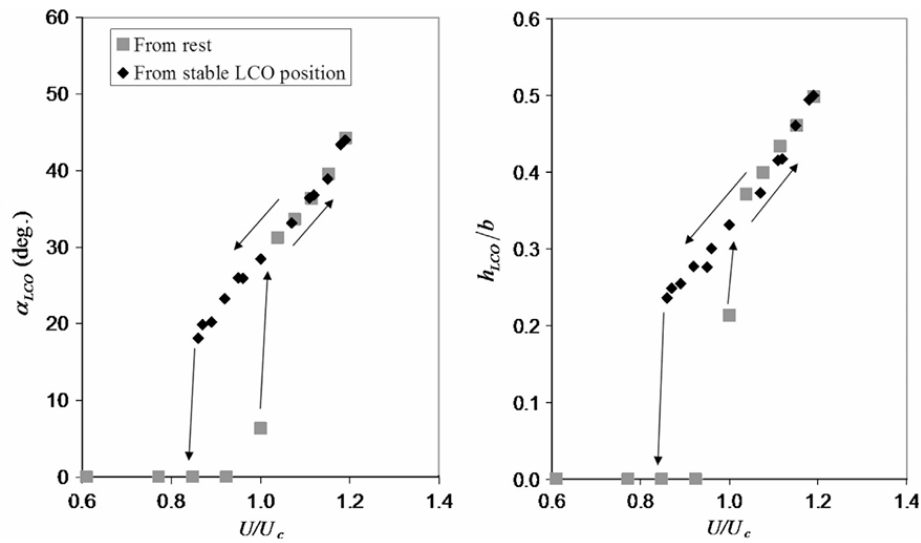


Figure 13 – LCO amplitude of the pitching and plunging response versus relative velocity. Source: Aman-dolese, Michelin and Choquel (2011).

To numerically evaluate the LCO behavior in both ascending and descending airspeed cases, a specific initialization procedure is adopted. Starting from low airspeeds, the system response is simulated from rest and integrated over a sufficiently long time interval to ensure convergence to a steady-state solution. As the airspeed is progressively increased, the simulations are repeated until the maximum airspeed is reached. Subsequently, the descending airspeed branch is simulated by using the final state vector of each previous simulation as the initial condition for the next one. From a numerical standpoint, this procedure is equivalent to applying an instantaneous step reduction in airspeed. This approach allows the identification and analysis of the hysteresis phenomenon observed in the LCO regime.

Finally, to specifically assess the aerodynamic implementation of the ONERA model, experimental data reported by Malher (2016) and shown in Figure 10 are used. These results allow the comparison of intermediate aerodynamic responses obtained with the present implementation for different imposed sinusoidal pitch frequencies, providing a focused validation of the unsteady aerodynamic behavior independently of the aeroelastic coupling.

RESULTS AND DISCUSSION

THIS chapter presents the main results obtained with the different aerodynamic and aeroelastic models implemented in this work. The results are discussed through a comparative analysis, highlighting the characteristics, advantages, and limitations of each method. The validation cases are based on the experimental data reported by Amandolese, Michelin and Choquel (2011) and described in Section 3.3.

The results are primarily organized according to the dynamic stall calibration, focusing first on the purely aerodynamic response of the models. Subsequently, the analysis is extended to the aeroelastic results, including the identification of the critical flutter velocity. The comparison then proceeds through the evaluation of modal frequencies as a function of airspeed, the amplitudes of the resulting LCOs, and, finally, the time-domain responses of the system.

4.1 Aerodynamic Tests

4.1.1 *ONERA model*

The ONERA aerodynamic model is implemented according Section 3.2.3. As an initial step, the model parameters were adopted from the values reported by Malher (2016) and are summarized in Table 1. These parameters provide a baseline representation of the aerodynamic response under dynamic stall conditions.

Nevertheless, discrepancies observed in the preliminary simulation results indicated the need for a recalibration of the model parameters in order to achieve closer agreement with the reference data presented in Figure 10. It is important to remark that differences arising from the use of the coefficients proposed by Malher (2016) are expected, since the model implemented in the present work follows the formulation of Dimitriadis (2017), which presents formulation differences in the differential equations.

This adjustment process was carried out manually, guided by a sensitivity analysis of the most influential empirical parameters. The final calibrated values were obtained by starting from the originally proposed parameters and progressively adjusting them according to the trends observed in the response curves, resulting in the updated parameter set presented in Table 5.

Table 5 – Aerodynamic calibrated coefficients for ONERA model.

k_l	s_l	λ_l	κ_l	a_{0_l}	a_{2_l}	r_{0_l}	r_{2_l}	σ_{0_l}	σ_{2_l}	E_{2_l}	$\Delta\tau_{d_l}$
0	2π	0.119	0.95	0.168	0.104	0.15	0.09	0.1	-0.005	0.005	$3b/U$

s_m	$\bar{\sigma}_m$	a_{0_m}	a_{2_m}	r_{0_m}	r_{2_m}	σ_{0_m}	σ_{2_m}	E_{2_m}	$\Delta\tau_{d_m}$
$-3\pi/16$	$-\pi/4$	0.2	0	0.19	0.03	-1	0.1	0	b/U

The validation case consists of an imposed sinusoidal pitching motion with an amplitude of 36° and three reduced frequencies, $k = 0.01$, 0.02 , and 0.03 . All other flow parameters are kept constant across the simulations. The final calibrated parameter set is as presented in Table 5 and the corresponding lift coefficient responses are shown in Fig. 14.

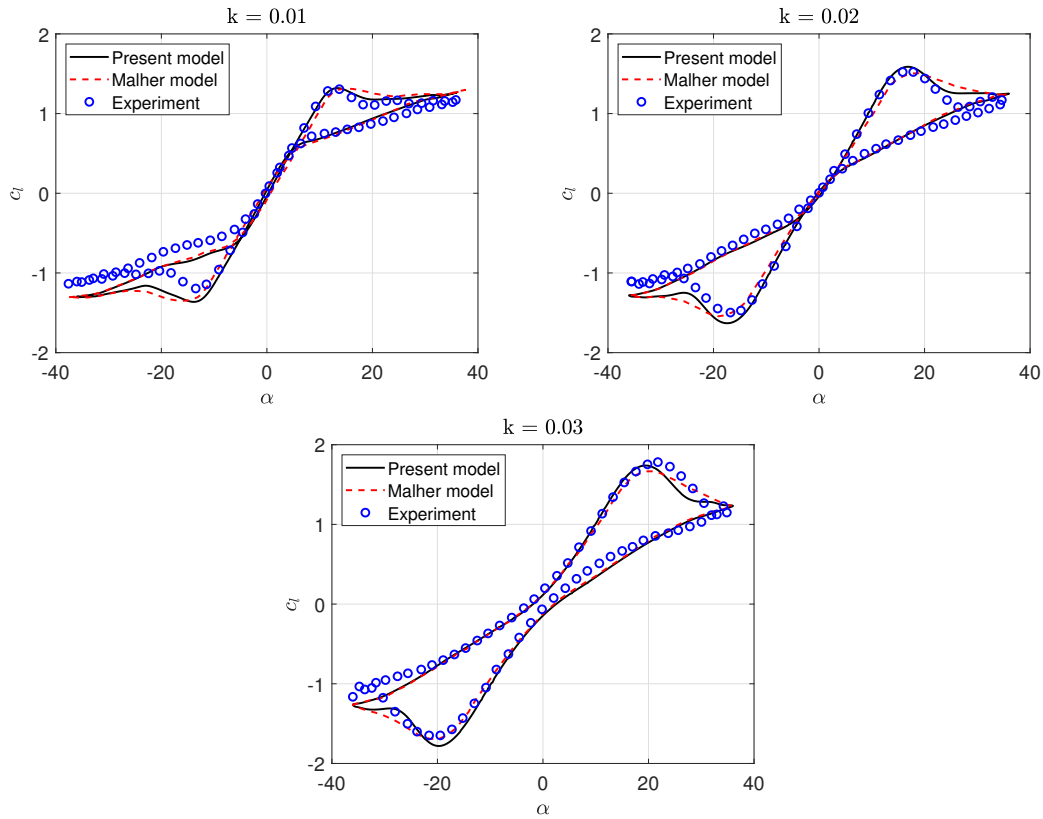


Figure 14 – ONERA lift coefficient compared to Malher (2016).

Overall, the lift coefficient c_l presents good agreement with the reference ONERA results from Malher (2016). Some discrepancies are observed in the negative angle-of-attack region, which can be attributed to asymmetrical aerodynamic effects in the experiment not fully captured by the model. For the lowest reduced frequency, $k = 0.01$, the lift response follows

a predominantly linear behavior prior to stall and after reattachment, as expected. At higher reduced frequencies, this linear region becomes less evident due to the increased influence of unsteady effects and non-uniform velocity distributions associated with higher pitch rates.

For the pitching moment coefficient $c_{m_{xf}}$, shown in Fig. 15, the overall trends are also reasonably captured, although larger discrepancies are observed when compared to the lift results. This behavior is consistent with the known limitations of the ONERA model in predicting pitching moments, as discussed by Dimitriadis (2017).

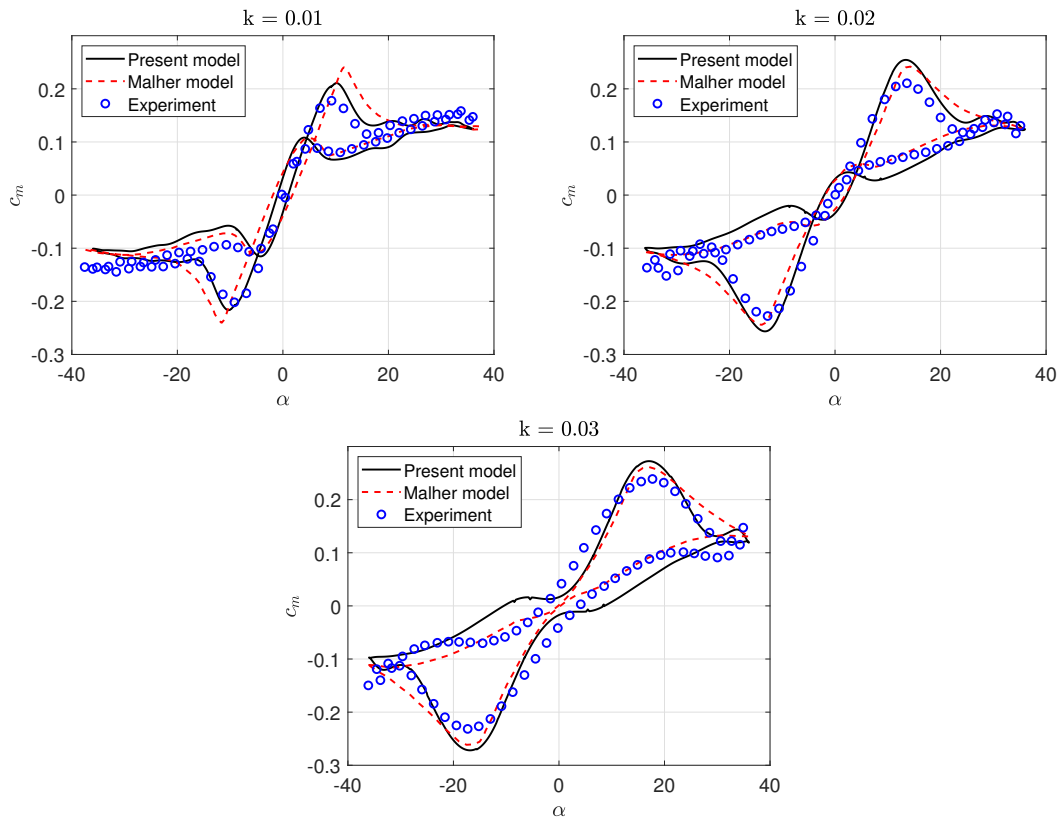


Figure 15 – ONERA pitch moment coefficient compared to Malher (MALHER, 2016).

In comparison with the example presented in Fig. 3, the pronounced moment peak associated with leading-edge vortex shedding is not clearly observed, indicating that this effect is likely weak or absent under the considered test conditions. Additionally, the enclosed area of the hysteresis loop increases with the reduced frequency, reflecting stronger unsteady effects. At the highest frequency, $k = 0.03$, the results exhibit increased uncertainty, suggesting reduced reliability of the model predictions in this regime.

After the initial aerodynamic validation, the ONERA model was subsequently coupled with the structural degrees of freedom to perform aeroelastic simulations, which are mainly described in the following subsections. The purpose of introducing this discussion at this step is to highlight that the calibration of ONERA model parameters may require an iterative procedure, incorporating feedback from the aeroelastic response.

In the first aeroelastic simulations, it was observed that the oscillation amplitudes increase rapidly after the critical airspeed, eventually saturating at pitch angles close to $\pm 90^\circ$. This behavior is not representative of the expected physical response. Additionally, the reduced frequencies obtained in these simulations were approximately $k \approx 0.09$, which is nearly three times higher than the maximum reduced frequency for which the model was originally calibrated using the experimental data reported by Malher (2016).

A plausible explanation for this behavior is the excessive energy input associated with the pitching moment coefficient $c_{m_{xf}}$ at higher reduced frequencies. As k increases, the internal area of the hysteresis loop of $c_{m_{xf}}$ grows significantly, indicating a larger amount of aerodynamic work being transferred to the structure. This effect can be quantified through the closed integral

$$W = \oint c_{m_{xf}} d\alpha, \quad (4.1)$$

where clockwise loops correspond to positive aerodynamic work and, consequently, to an amplification of the structural response.

In order to obtain a more suitable aeroelastic response, the most influential ONERA parameters were modified based on the sensitivity analysis presented by Malher (2016). Improved results, characterized by reduced oscillation amplitudes, were obtained by adjusting the parameters summarized in Table 6.

Table 6 – Adjusted ONERA parameters adopted for aeroelastic simulations.

κ_l	a_{0_l}	E_{2_l}	σ_{2_l}
0.74	0.12	0	-0.15

Nevertheless, even with these adjustments, the resulting limit-cycle oscillation amplitudes obtained from the bifurcation analysis still present noticeable discrepancies when compared with the experimental reference data.

It is also noteworthy that modifications applied exclusively to lift-related parameters have a significant influence on the pitching response, highlighting the strong coupling between lift and moment in the aeroelastic system. Conversely, variations applied only to moment-related parameters did not lead to improved bifurcation characteristics.

Ideally, a systematic optimization of the ONERA model parameters based on the complete aeroelastic bifurcation curve would provide a more robust calibration strategy, allowing the influence of each parameter to be evaluated as a function of airspeed. However, such an approach was not pursued in this work due to the increased implementation complexity and computational cost involved, with individual simulations requiring on the order of tens of seconds to complete.

4.2 Aeroelastic Tests

The following discussion presents a comparative analysis of the different models in terms of their aeroelastic response, considering free pitch and plunge degrees of freedom.

4.2.1 Critical Velocity

The critical velocity U_c corresponds to the airspeed at which the aeroelastic system loses stability and the onset of LCOs is observed. Table 7 summarizes the comparison between the implemented methods for identifying U_c and the corresponding critical frequency ω_c , using the experimental results reported by Amandolese, Michelin and Choquel (2011) as reference.

Before presenting a direct comparison, it is important to highlight that, according to Amandolese, Michelin and Choquel (2011), the critical velocity is primarily associated with the linear, attached-flow characteristics of the aerodynamic response. As a result, estimates obtained using quasi-steady and unsteady attached-flow models, such as Wagner's formulation, remain relevant and meaningful for comparison. Beyond this airspeed, models that do not account for stall effects may exhibit responses going towards the infinite, failing to capture sustained LCO behavior at higher velocities.

Table 7 – Results obtained for the critical velocity and frequency.

	Amandolese (Experiment)	Amandolese (Theodorsen)	Quasi-steady	Wagner	ONERA	Leishman– Beddoes
$U_c/b\omega_\alpha$	10.6	9.8	10.1	9.9	9.5	10.1
ω_c [rad/s]	48.1	–	51.3	45.5	45.9	46.5

The quasi-steady model shows good agreement with the experimental critical velocity, reinforcing the observation that flutter onset is largely governed by linear aerodynamic behavior. The estimated critical frequency, obtained from the system poles, is also close to the experimental value, although it is slightly higher and represents the largest deviation among the methods considered.

For the Wagner unsteady aerodynamic model, the predicted critical velocity remains close to both the experimental and Theodorsen-based results, consistent with their theoretical relationship. The corresponding critical frequency is slightly lower than the experimental value, reflecting the influence of aerodynamic memory effects inherent to the model, as opposed to the instantaneous response assumed in the quasi-steady formulation. Despite its inability to represent stall effects, Wagner's model proves to be a useful tool for initial flutter estimation.

In the case of the ONERA model, the results indicate an earlier flutter onset when compared to the other methods, suggesting a strong influence of the fitted aerodynamic coefficients. Nevertheless, the predicted critical frequency remains coherent with the remaining models and slightly below the experimental value.

The Leishman–Beddoes model provides the closest estimates for both the critical velocity and frequency. This result can be attributed to its more comprehensive representation of the aerodynamic phenomena, which contribute to a more accurate prediction of the aeroelastic instability onset.

4.2.2 Modes and Frequencies

The evolution of the pitch and plunge modes in terms of their natural frequencies and the real part of the system poles is analyzed in Figure 16. Before the critical velocity U_c , the system response is characterized by low-amplitude oscillations and is predominantly governed by linear aeroelastic behavior. After U_c , the response transitions to a LCO, which becomes the dominant dynamic feature.

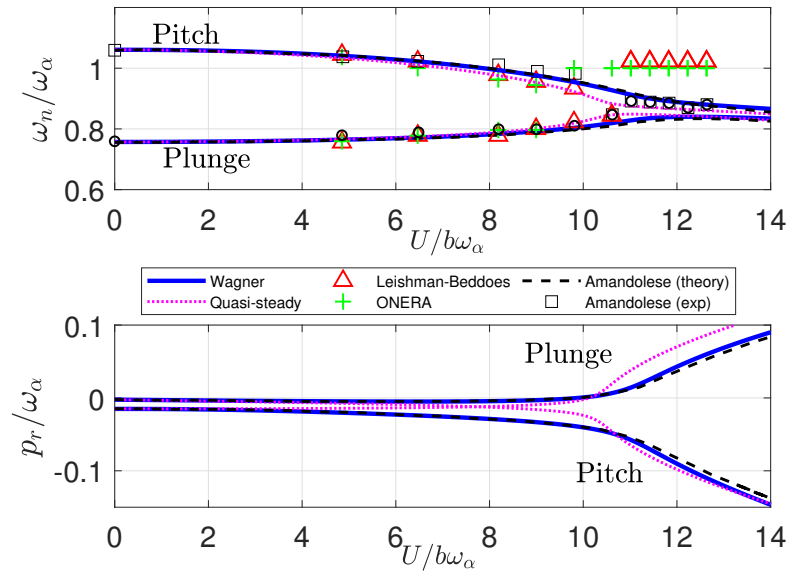


Figure 16 – Comparison of natural frequencies (ω_n) and real part of the poles (p_r) with the results of Amandolese, Michelin and Choquel (2011).

Experimental results show that, for airspeeds below U_c , the pitch and plunge modes exhibit clearly distinct natural frequencies. At the critical velocity, these two modes merge into a single oscillation characterized by a unique dominant frequency, a behavior that persists for higher airspeeds. This resulting frequency is between the two linear natural frequencies predicted by the Theodorsen-based model employed by Amandolese, Michelin and Choquel (2011), as expected from classical flutter theory.

An interesting observation is that the pitch-dominated mode presents a natural frequency approximately 5% higher than the isolated pitch frequency even at zero airspeed. This behavior is explained by the presence of the inertial coupling term S in the structural inertia matrix, which introduces coupling between pitch and plunge motions and prevents them from being dynamically independent.

The unsteady Wagner formulation presents results equivalent to those obtained with Theodorsen's method, as expected from their theoretical relationship in the attached-flow regime. The quasi-steady model also shows a frequency evolution close to the experimental reference, although slightly larger discrepancies are observed for the pitch mode near the critical airspeed. In terms of the real part of the poles, more significant deviations appear above U_c , which are associated with the growth of oscillation amplitudes and the absence of nonlinear aerodynamic effects in the model.

For the ONERA and Leishman–Beddoes models, the predicted modal frequencies below the critical velocity remain very close to both the theoretical and experimental values. However, for airspeeds above U_c , both models exhibit a tendency for the dominant frequency to approach the isolated pitch natural frequency ω_α , a behavior that is not observed experimentally. This suggests a numerical or modeling limitation related to the dominance of pitch motion over plunge in the post-critical regime, which is also seen in the next amplitude analysis. Nevertheless, at the critical velocity itself, both models correctly predict the frequency values reported in Table 7, before converging toward ω_α at higher amplitudes. This transition is also evident in the amplitude–frequency diagram obtained from the Fourier transform, in which the peak at ω_c gradually decreases while a secondary peak at ω_α increases, until it becomes dominant and the prevailing frequency shifts. Among the nonlinear aerodynamic models, the Leishman–Beddoes formulation provides the most coherent representation of the experimental behavior at U_c .

4.2.3 Amplitudes of the LCO

In this part, the amplitudes of the LCO are discussed. Table 8 presents the LCO amplitudes at the critical velocity, compared with the experimental reference results. For the pitch response, all numerical models predict amplitudes larger than those observed experimentally. Among them, the quasi-steady model shows the largest deviation, which can be attributed to its simplified aerodynamic representation. The remaining models, which include some form of unsteady aerodynamic effects, yield pitch amplitudes closer to the experimental range. In particular, the Leishman–Beddoes model provides the best agreement, with values fully within the experimental uncertainty bounds.

Table 8 – Pitch and plunge LCO amplitudes at the critical velocity U_c .

	Amandolese (Experiment)	Quasi-steady	Wagner	ONERA	Leishman– Beddoes
$\alpha_{\text{LCO}} [^\circ]$	6.5 ± 0.5	14.5	8.2	7.4	6.7
h_{LCO}/b	0.22 ± 0.04	0.062	0.18	0.12	0.10

For the plunge response, all models underestimate the oscillation amplitude when compared to the experimental measurements. In this case, the Wagner model yields the closest prediction. The larger discrepancies observed in plunge are likely related to differences in the

predicted lift forces, since plunge motion is predominantly driven by the vertical aerodynamic load. Conversely, the pitch response is more strongly influenced by the aerodynamic moment modeling, which explains the comparatively better agreement observed for pitch amplitudes in models with a more detailed moment representation.

For the aerodynamic models that include stall representation, the evolution of oscillation amplitudes as a function of airspeed can be evaluated. Figure 17 shows the bifurcation results obtained with the ONERA aerodynamic model. In this case, the predicted plunge amplitudes are smaller than the experimental reference, while the pitch amplitudes are comparatively higher. This behavior indicates deficiencies in the representation of both lift and pitching moment under these conditions. As discussed by Malher (2016), the amplitude evolution with airspeed is strongly dependent on the calibration of the empirical parameters, which still leaves room for further improvement.

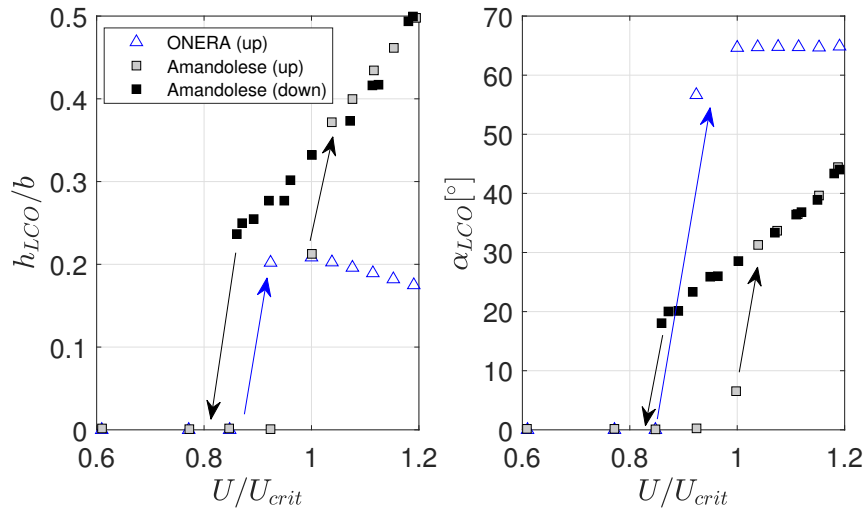


Figure 17 – ONERA bifurcation compared to Amandolese, Michelin and Choquel (2011).

The bifurcation comparison for the Leishman–Beddoes model is shown in Figure 18. The hysteretic behavior is well captured, with a partial growth of oscillation amplitude at U_c during the ascending branch and sustained oscillations persisting down to approximately $0.88U_c$ during the descending branch. Since the numerically simulated airspeed is held constant during each simulation, the final state of a given case is used as the initial condition for the subsequent airspeed step.

The predicted plunge amplitudes are approximately two to three times smaller than the experimental values, indicating limitations in the lift modeling, likely associated with the aerodynamic validation being restricted to pitch amplitudes below 30° . In contrast, the predicted pitch amplitudes are closer to the experimental reference, with maximum deviations of about 10° , suggesting a more accurate representation of the pitching moment coefficient c_m .

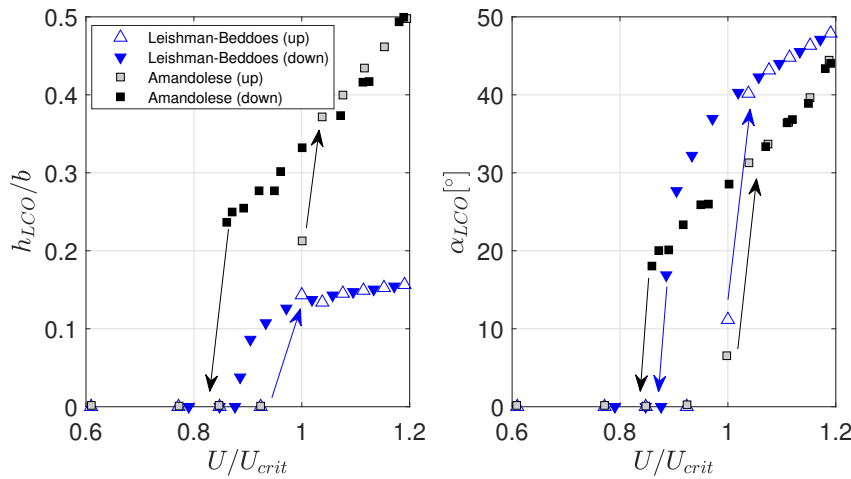


Figure 18 – Leishman–Beddoes bifurcation compared to Amandolese, Michelin and Choquel (2011).

4.2.4 Temporal Simulation

Finally, an additional comparison of interest concerns the temporal evolution of the aeroelastic response and its agreement with the experimental measurements. Focusing on the post-critical regime, where LCOs are present, the Leishman–Beddoes model is considered, as it provides the most consistent results among the implemented formulations. It is worth recalling that the quasi-steady and Wagner models assume attached flow, being unable to capture dynamic stall effects or to represent LCOs beyond the critical speed, leading instead to a continuously growing unstable response.

The simulations are performed with an initial plunge displacement of $0.18h/b$, while the pitch angle and its derivatives are initially set to zero, as adopted by Amandolese, Michelin and Choquel (2011). Figure 19 presents a temporal comparison between the numerical and experimental responses at the critical velocity U_c predicted by the Leishman–Beddoes model. In the experiment, the pitch amplitude exhibits a gradual growth before reaching its LCO value, whereas the numerical simulation shows an initial transient with peak amplitudes before converging to a similar LCO amplitude and within approximately the same convergence time. For the plunge response, the experiment displays an initial amplitude decay followed by growth toward the LCO amplitude, with a noticeable periodic oscillation of the amplitude, while the numerical simulation converges to a smaller LCO amplitude after a similar initial decay. Since the equivalent pitch angles do not exceed the static stall angle, dynamic stall effects are not observed under these conditions.

Figure 20 presents a temporal comparison between the numerical and experimental responses at an airspeed of $1.08U_c$, which ensures oscillation amplitudes characteristic of stall flutter. In the experiment, the pitch amplitude again exhibits a gradual increase before reaching its steady-state LCO value, whereas the numerical model converges more rapidly to a similar amplitude. For the plunge response, both the experiment and the numerical simulation show an

initial low-amplitude phase followed by a rapid growth in oscillation amplitude. However, the numerical model predicts a more abrupt transition, which may indicate variations in aerodynamic behavior not fully captured by the model formulation, particularly during the onset of dynamic stall.

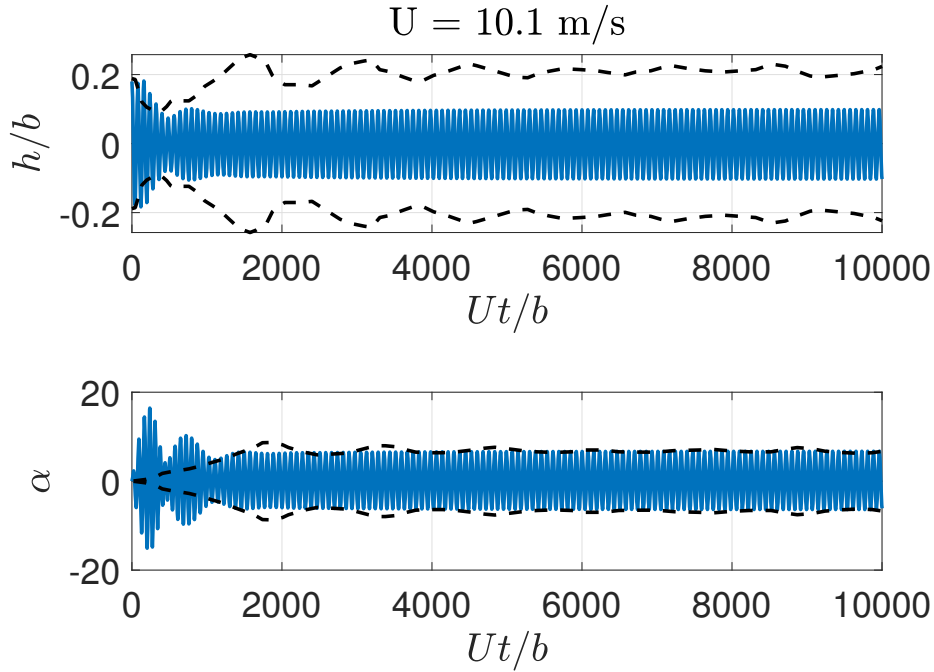


Figure 19 – Temporal response obtained with the Leishman–Beddoes model (solid line) compared to Amandolese, Michelin and Choquel (2011) (dashed line) at U_c .

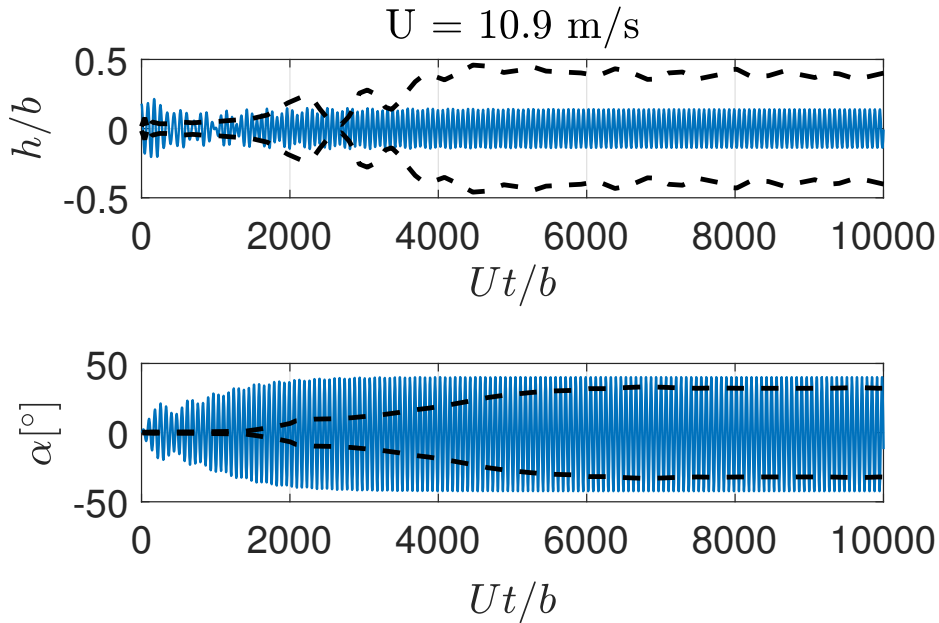


Figure 20 – Temporal response obtained with the Leishman–Beddoes model (solid line) compared to Amandolese, Michelin and Choquel (2011) (dashed line) at $1.08 U_c$.

4.3 Overall Discussion

As a summary of the observed results, each aerodynamic model exhibited different performance characteristics in the aeroelastic analyses. Among the attached-flow formulations without stall representation, a significant difference is observed between the quasi-steady model, which assumes instantaneous aerodynamic response, and the unsteady Wagner model. Both approaches are shown to be suitable for identifying the flutter phenomenon up to the critical velocity U_c . However, their predictions beyond this point are not consistent with the nonlinear behavior associated with stall and LCOs. Between these two models, the Wagner formulation is therefore the more reliable option.

Regarding the ONERA model, its reliability is strongly dependent on accurate parameter calibration. This calibration can be achieved either by adopting parameters previously adjusted in the literature or by fitting the model directly to experimental data under representative conditions, as emphasized by Dimitriadis (2017). In the latter case, the required experimental inputs include measurements of $c_l(\alpha)$ and $c_m(\alpha)$ under both static and imposed dynamic conditions, enabling subsequent free-response aeroelastic predictions.

Overall, the Leishman–Beddoes model provided the most satisfactory results among the nonlinear aerodynamic formulations considered. This performance is largely attributed to the extensive validation of the model reported in the literature, particularly by Santos and Marques (2021b). In this work, it proved to be the most reliable approach for predicting the critical velocity, oscillation frequency, and LCO amplitudes. Beyond the critical speed, it was also the only model capable of producing reasonably consistent results in terms of amplitude evolution and temporal response, although the dominant frequency in this regime still tended to coincide with the expected structural pitch frequency.

For future improvements, it is recommended to employ the same airfoil geometry (flat plate) and to operate within smaller oscillation amplitudes, respecting the commonly adopted upper limit of 30° . One possible strategy is to define the structural equilibrium about a positive mean pitch angle, which typically induces flow separation on only one side of the airfoil and leads to reduced oscillation amplitudes. Additionally, since the aerodynamic validation is limited to Mach numbers above approximately 0.035, operating at airspeeds above this threshold is advised. Finally, the computational cost of the nonlinear models remains comparable to that of the previous formulations, with simulation times on the order of tens of seconds.

CONCLUSIONS

AFTER the previous presentation of the developed work, starting from the bibliographic review, passing through the model adaptations, and ending in the obtained results, this chapter aims to summarize the main conclusions, key learnings, and perspectives for future related works.

5.1 Conclusion

Taking into account the experimental reference case of Amandolese, Michelin and Choquel (2011), as well as the possibility of future experiments at ISAE–SUPAERO, this work aimed to offer multiple aerodynamic modeling tools and to evaluate the different methods in terms of their agreement with the expected aeroelastic behavior. From this comparative analysis, it is possible to draw several indications regarding the most suitable approaches for stall flutter investigations.

Among the methods assuming attached flow, the Wagner unsteady aerodynamic model stands out as the most recommended. It is theoretically linked to the Theodorsen formulation in the frequency domain and its time-domain implementation provides closer estimations of the critical velocity, LCO frequency, and amplitudes when compared to the experimental results. This behavior is explained by the fact that flutter onset is mainly governed by the linear part of the aerodynamic model, which also justifies the global proximity of the results among all the formulations.

As a consequence, the Wagner model, despite its relatively low complexity, is capable of providing important insights for planning future experimental campaigns. In particular, it allows the identification of suitable combinations of structural dimensions, elasticity, damping, and inertia that are coherent with wind tunnel limitations in terms of airspeed and available space. For these reasons, it is a convenient tool for adjusting structural constants and estimating the

flutter critical speed. Nevertheless, coherent predictions beyond the critical velocity U_c are not possible with this model, since stall nonlinear effects are not included. When compared to the quasi-steady formulation, the Wagner model clearly offers more accurate information, as the instantaneous aerodynamic assumptions of the quasi-steady approach become less coherent for stall flutter phenomena, which usually occur at relatively high frequencies.

When the objective of the theoretical analysis is to predict and analyze the system behavior above the critical speed, semi-empirical dynamic stall modeling becomes necessary due to the large oscillation amplitudes and the introduction of nonlinear stall effects. Since this phenomenon is not fully described by the flow-attached theories, these models inherently depend on calibration with experimental reference data. The ONERA model represents the aerodynamic coefficients as functions of circulation states governed by differential equations. The calibration performed in this work, aiming to reproduce the results of Malher (2016) (shown in Figs. 14 and 15), demonstrated that the model is able to capture the expected aerodynamic behavior under imposed sinusoidal oscillations. However, the free-motion aeroelastic results (Fig. 17) still show considerable discrepancies in amplitude, likely due to operating at reduced frequencies higher than those used during calibration. It is therefore expected that the representation of the bifurcation behavior would improve if the parameters were optimized over a frequency range closer to the flutter condition. For future developments it is recommended optimization based on representative experimental data, either from the literature or from wind tunnel tests, requiring airload coefficients measured under both static and imposed dynamic conditions.

The Leishman–Beddoes model computes the individual contributions of the dynamic stall mechanisms separately. By adopting reference parameters previously validated for a given airfoil, as reported by Santos and Marques (2021b), it was possible to obtain more representative bifurcation results, as illustrated in Fig. 18. The hysteretic behavior is well reproduced, and the pitch amplitudes follow a growth trend with airspeed that is closer to the experimental observations. The temporal simulations also provided interesting qualitative agreement with the measured response. For these reasons, among all the tools implemented in this work, the Leishman–Beddoes model is recommended as an intermediate-accuracy approach for capturing the general aeroelastic behavior above U_c . For improved results, it is suggested to operate within conditions consistent with the model validation, in terms of airfoil geometry, oscillation amplitudes, and airspeed. Overall, the Leishman–Beddoes model proved to be the most suitable for estimating flutter amplitudes in dynamic stall conditions, with computational times on the order of tens of seconds per simulation.

In summary, the Wagner model is well suited for simpler and more generic flat-plate estimations up to the flutter onset, while the Leishman–Beddoes formulation is more appropriate for analyses beyond the critical speed. It is also reinforced that the ONERA model can be accurate and reliable in many applications, as demonstrated by Malher (2016), but it requires sufficient and representative data for prior aerodynamic calibration.

5.2 Future Works

If the objective is to further exploit the methods implemented in this work, several paths can be considered. As described by Dimitriadis (2017), both quasi-steady and unsteady formulations can be extended to include a control surface at the trailing edge of the plate. Such formulations are particularly useful for control applications, as they can directly influence the global characteristics of LCOs.

For the semi-empirical approaches, future work should focus on using additional validation cases and performing systematic optimization of the aerodynamic parameters. This would allow the development of more robust models and increase the predictive accuracy of both the ONERA and Leishman–Beddoes formulations.

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